The generalized 3-connectivity of equally complete k-partite graph and its line graph

Dheyaa D. Kadhim Alaa A. Najim

Department of Mathematics, College of Science, University of Basrah

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Abstract

For a vertex set S with cardinality at least 2 in a graph G, we need a tree in order to connected the set, where this tree is usually called a Steiner tree connecting S (or an $S-$ tree). Two Steiner trees T and T' are said to be internally disjoint if $V(T)\bigcap V(T')=S$ and $E(T)\bigcap E(T')=\phi$. Let $\kappa_G(S)$ denote the maximum number of internally disjoint Steiner trees connecting S in G. The generalized k-connectivity $\kappa_k(G)$ of a graph G which was introduced by Chartand et al. (1984) and defined as: $\kappa_k(G)=\min\{\kappa_G(S):S\subseteq V(G)$ and $|S|=k\}$. In this paper we determine the generalized 3-connectivity of equally complete k-partite graph and its line graphs.

Keywords : The generalized 3- connectivity, internally disjoint trees, Steiner trees, the line graph, the complete k-partite graph.

1. Introduction

The graphs in this paper are simple and undirected. For a graph G, the set of vertices, the set of edges, and the line graph of G are denoted by $V(G)$, $E(G)$, and $L(G)$ respectively. The generalized connectivity of a graph G which introduced by Chartrand et al. in[1], is a natural and nice generalization of the vertex connectivity.

The connectivity of the graph G is defined as $\kappa(G)=\min\{\kappa_G(S):S\subseteq V,\ |S|=2\}$, where $\kappa_G(S)$ is the maximum number of internally disjoint paths from u to v in G $(S=\{u,v\})$ [2,3]. The subgraph $T=(V',E')$ of the graph $G=(V,E)$ is called Steiner tree connecting S (S-tree) if T is a tree and the set $S\subseteq V$ of at least two vertices. Two Steiner trees T and T' connecting S are said to be internally disjoint if $E(T)\bigcap E(T')=\phi$ and $V(T)\bigcap V(T')=S$. For an integer k with $2\le k\le |V(G)|$ and $S\subseteq V(G)$ with $|S|\ge 2$, the generalized k-connectivity is define as: $\kappa_k(G)=\min\{\kappa_G(S):S\subseteq V(G),\ |S|=k\}$,where $\kappa_G(S)$ is maximum number of internally disjoint Steiner trees connecting S in G. Clearly $\kappa_2(G)=\kappa(G)$. In [4] Li, Shasha, Wei Li, and Xueliang Li, determined the generalized connectivity of complete bipartite graphs. Later, they di-cussed the generalized connectivity of the complete equipartition 3-partite graphs in [5], the generalized 3-connectivity of graph products [3], also see[2,6,7,8,9].

Let $K_k(n)$ be an equally complete k -partite graph with partition $\{X_1, X_2, ..., X_k\}$ where $|X_i| = n,\ i = 1,2,...,k,$ and let $H_{ij} = G\left[X_{ij}\right],\ i = 1,2,...,k,\ j = 1,2,...,n$ be an induced subgraph of G by the set $X_{ij} = \{x_{rs} \in X_r : r = 1,2,...,k,\ s = 1,2,...,n,\ r \neq i\} \bigcup \{x_{ij}\}$, see figure (1($a$)). The line graph $L(K_k(n))$ of the equally complete k-partite graph is a graph that $V(L(K_k(n)) = E(K_k(n))$ and two vertices $u = u_1u_2,\ v = v_1v_2 \in V(L(K_k(n)))$ are adjacent if either $(u_1 = v_1$ and $u_2v_2 \in E(K_k(n))$ or $\left(u_2 = v_2 \ and\ u_1v_1 \in E(K_k(u))\right)$. The line graph $L(H_{ij})$ of the star graph H_{ij} is complete graph of order $(k-1)n+1$ with $V(L(H_{ij})) = \{x_{ij}x_{rs} : r = 1,2,...,k, s = 1,2,...,n, r \neq i\}$ see figure (1(b)).

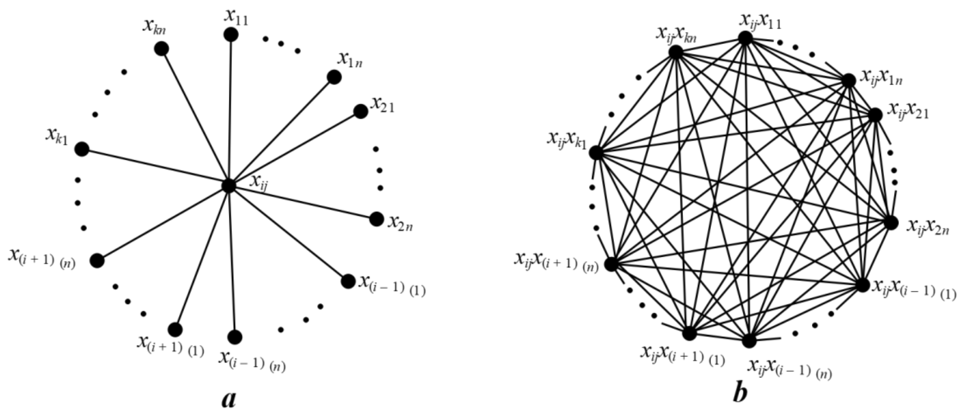

Figure 1: $(a)\, H_{ij}$ and $(b)\, L(H_{ij})$

2. Preliminary results

Proposition 2.1 [10] Two simple graphs G and H are isomorphic if and only if there is a bijective mapping. $\theta : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\theta(u)\theta(v) \in E(G)$.

Proposition 2.2 [9] Let G be a connected graph of order u with minimum degree δ. If there are two adjacent vertices of degree δ, then $\kappa_k(G) \leq \delta - 1$ for $3 \leq k \leq n$. Moreover, the upper bound is sharp.

3. Main results

In this section, we determine the value of the generalized 3-connectivity of the equally complete k -partite graph and its line graph. First we introduce two lemmas that are important to the main results.

Lemma 3.1 For any two positive integers k and n, $K_k(n) \cong \bigcup_{i=1}^{k}\left(\bigcup_{j=1}^{n} H_{ij}\right)$.

Proof: Let $G = K_k(n)$ and $H = \bigcup_{i=1}^{k}(\bigcup_{j=1}^{n}(H_{ij}))$. $|V(G)| = kn$, $|V(H)| = \frac{1}{1+n(k-1)}\sum_{i=1}^{k}\sum_{j=1}^{n}|V(H_{ij})|$ $= \frac{1}{1+n(k-1)}\sum_{i=1}^{k}\sum_{j=1}^{n}(1+n(k-1)) = nk$, then $|V(G)| = |V(H)|$. $|E(G)| = \frac{1}{2}k(k-1)n^2$, $|E(H)| =$ $\sum_{i=1}^{k}\sum_{j=1}^{n}|E(H_{ij})| = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{(k-1)n}{2}\right) = \frac{1}{2}k(k-1)n^2$, then $|E(G)| = |E(H)|$.

Define $f: V(G) \to V(H)$, as $f(x) = x, \forall x \in V(G)$,then f is bijective mapping. Let $uv \in E(G)$, then there are $i, i' = 1, 2, ..., n$, $i \neq i'$ such that $u \in X_i, v \in X_{i'}$, also there are $j, j' = 1, 2, ..., k$ such that $u = x_{ij} \in X_{ij}$, $v = x_{i'j'} \in X_{i'j'}$. Clearly, there exist $uv = x_{ij}x_{i'j'} \in E(H_{ij}) \bigcap E(H_{i'j'}) \subseteq E(H)$ for some $i, i' = 1, 2, ..., n$ and $j, j' = 1, 2, ..., m$. Since $f(u)f(v) = uv$, then $f(u)f(v) \in E(H)$ from proposition (2.1). ■

Lemma 3.2 For any two positive integers k and n, $L(K_k(n)) \cong \bigcup_{i=1}^{k}\left(\bigcup_{j=1}^{n} L(H_{ij})\right)$.

Proof: Let $L(G) = L(K_k(n))$ and $L(H) = \bigcup_{i=1}^{k}\bigcup_{j=1}^{n} L(H_{ij})$. $|V(L(G))| = \frac{1}{2}k(k-1)n^2$, $|V(L(H))|$ $\sum_{i=1}^{k}\sum_{j=1}^{n}|V(L(H_{ij}))| = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{(k-1)n}{2}\right) = \frac{1}{2}k(k-1)n^2$, then $|V(L(G))| = |V(L(H))|$. $|E(L(G))| = \frac{1}{2}$ $\sum_{i=1}^{k}\sum_{j=1}^{n}(d_G(x_{ij}))^2 - E(G(x_{ij})) = \frac{1}{2}k(k-1)^2n^3 - \frac{1}{2}k(k-1)n^2 = \frac{1}{2}k(k-1)n^2((k-1)n-1),$

$|E(L(H))| = \sum_{i=1}^{k}\sum_{j=1}^{n}|E(L(X_{ij})| = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{(k-1)n((k-1)n-1}{2}\right) = \frac{1}{2}k(k-1)n^2((k-1)n-1)$,then $|E(L(G))| = |E(L(H))|$, see figure(2).

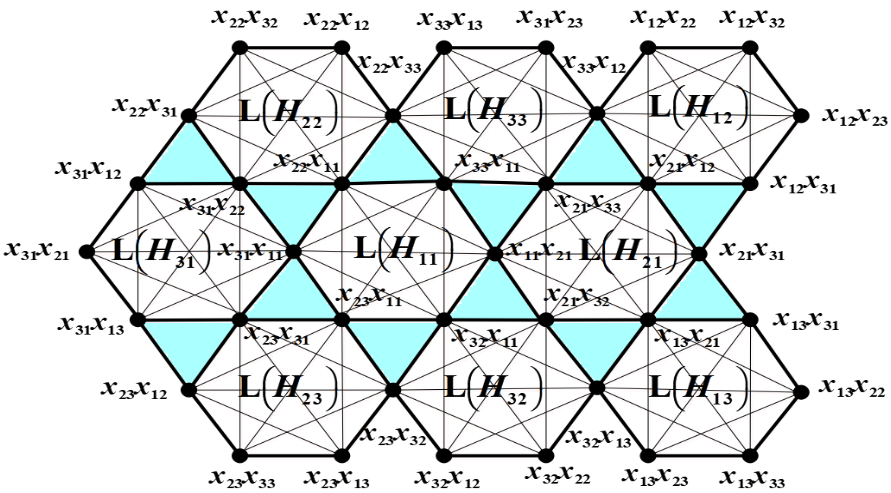

Figure 2: $L(K_3(3))$

Suppose $a \in V(L(K_k(n))$, this mean a is an edge in $K_k(n)$, thus there are two vertices $x_{ij}, x_{i'j'}$ in $K_k(n), \forall i \neq i'$ such that $a = x_{ij}x_{i'j'}$. Then $a \in E(H_{ij}) \bigcap E(H_{i'j'})$ for some $i, i' = 1, 2, ..., k$ and $j, j' = 1, 2, ..., n$, *i.e* $a \in V(L(H_{ij})) \bigcap V(L(H_{i'j'}))$, thus $a \in V(L(H))$.

Define $f : V(L(G)) \to V(L(H))$, as $f(x) = x, \forall x \in V(L(G))$, then f is bijective mapping. Let $ab \in E(L(G))$ such that $a = x_{ij}x_{i'j'}$, $b = x_{ij}x_{i''j''}, \forall i, i', i'' = 1, 2, ..., k,$ $i \neq i', i \neq i''$ and $j, j', j'' =$ $1, 2, ..., n,$ such that $a \in X_{ij}$, $b \in X_{i'j'}$. Clearly, there exist $ab = x_{ij}x_{i'j'}x_{ij}x_{i'j'} \in E(L(H_{ij})) \bigcap$ $E(L(H_{i'j'})) \subseteq E(L(H))$. Since $f(a)f(b) = ab$, then $f(a)f(b) \in E(L(H))$ from pro. (2.1). ■

Theorem 3.3 : Let $K_k(n)$ be a complete k-partite graph with two integers $k, n \geq 3$, the generalized 3-connectivity of $K_k(n)$ is $\left\lfloor \frac{(2k-3)n}{2} \right\rfloor \leq \kappa_3(K_k(n)) \leq n(k-1) - 1$.

Proof : Let $G = K_k(n)$ and $H = \bigcup_{i=1}^{k} \left(\bigcup_{j=1}^{n} H_{ij} \right)$, from the lemma (3.1) we have $G \cong H$. Since the degree of any vertex in H is $n(k-1)$, then H is $n(k-1)$-regular graph, by the proposition (2.2) we have $\kappa_3(H) \leq n(k-1) - 1$. For the completing the proof we just need to show that for any 3-subset $S = \{u, v, w\} \subseteq V(G)$, there exist at least $\left\lfloor \frac{(2k-3)n}{2} \right\rfloor$ internally disjoint Steiner trees connecting S in H. Since $H_{1j} = H_{2j} = ... = H_{ij} = K_{1,(k-1)n}$, $\forall i = 1, 2, ..., k$, $j = 1, 2, ..., n$, H_{1j} $\cong H_{2j} \cong ... \cong H_{ij}$, then we have three cases :

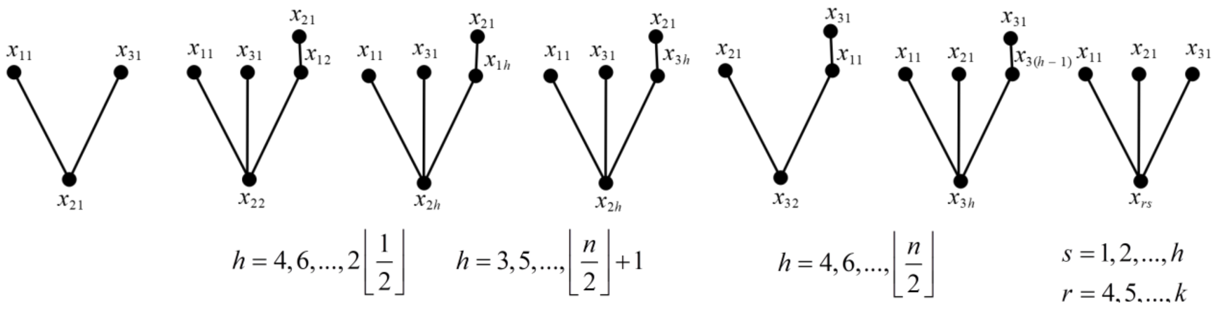

Figure 3

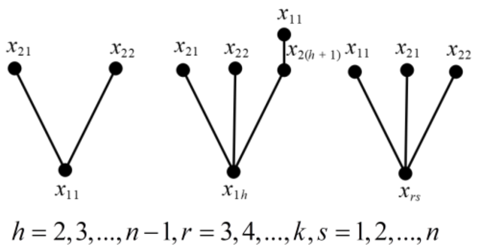

Figure 4

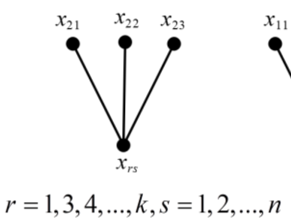

Figure 5

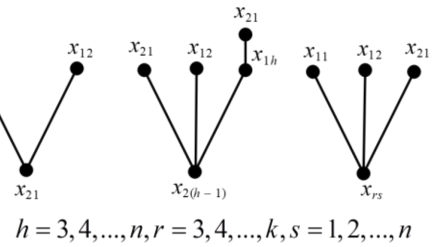

Figure 6

Case 1. If $u, v, w \in H_{ij}, \forall i = 1,2,...,k, j = 1,2,...,n$. Without loss of generality, we may put $i = 1$, $j = 1$, such that $u, v, w \in H_{11}$. Then there are three subcases:

Subcase (1.1) Let $u = x_{11}, v = x_{21}, w = x_{31}$. Then the maximum number of internally disjoint S-trees connecting S in H is $\left\lfloor \frac{(2k-3)n}{2} \right\rfloor$, see figure 3.

Subcase (1.2) Let $u = x_{11}, v = x_{21}, w = x_{22}$. Then the maximum number of internally disjoint S-trees connecting S in H is $(n(k-1)-1)$, see figure 4.

Subcase (1.3) Let $u = x_{21}, v = x_{22}, w = x_{23}$. Then the maximum number of internally disjoint S-trees connecting S in H is $(n(k-1)-1)$, see figure 5.

Case 2. If $u, v \in H_{ij}, w \notin H_{ij}, \forall i = 1,2,...,k , j = 1,2,...,n$. Again, we may assume $i = 1, j = 1$ such that $u, v \in H_{11}$ and $w \notin H_{11}$. Then there are two subcases :

Subcase (2.1) Let $u = x_{11}, v = x_{21}, w = x_{12}$.Then the maximum number of internally disjoint S-trees connecting S in H is $\left(n(k-1)-1\right)$, see figure 6.

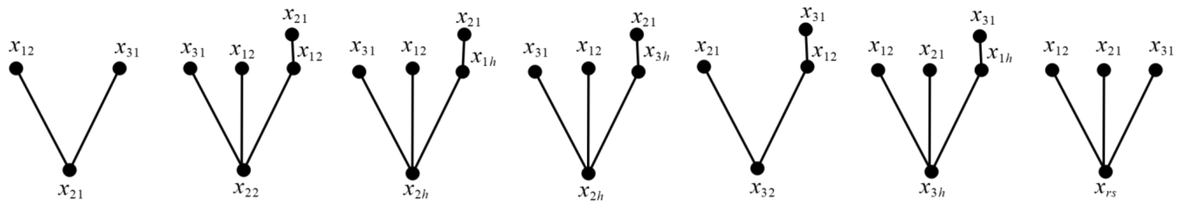

Figure: 7 $h=4,6,\dots,2\left\lfloor \frac{n}{2}\right\rfloor$ $h=3,5,\dots,2\left\lfloor \frac{n}{2}\right\rfloor+1$ $h=4,6,\dots,2\left\lfloor \frac{n}{2}\right\rfloor$ $s=1,2,\dots,n$ $r=4,5,\dots,k$

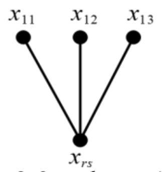

Figure 8: $r=2,3,\dots,k,\ s=1,2,\dots,n$

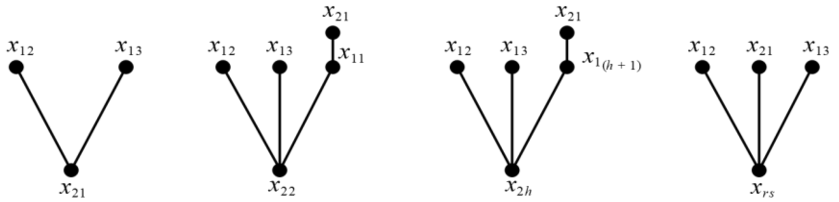

Figure 9: $h=3,4,\dots,n-1,\ r=3,4,\dots,k, s=1,2,\dots,n$

Subcase (2.2) Let $u=x_{21}, v=x_{31}, w=x_{12}$. Then the maximum number of internally disjoint S-trees connecting S in H is $\left\lfloor \frac{(2k-3)n}{2}\right\rfloor$, see figure 7.

Case 3. If $u\in H_{ij},\ v,w\notin H_{ij},\ \forall i=1,2,\dots,k,\ j=1,2,\dots,n.$ Assume $i=1, j=1$ such that $u\in H_{11}$, $v,w\notin H_{11}$. Then there are two subcases:

Subcase (3.1) Let $u=x_{11}, v=x_{12}, w=x_{13}$. Then the maximum number of internally disjoint S-trees connecting S in H is $(n(k-1))$, see figure 8.

Subcase (3.2) Let $u=x_{21}, v=x_{12}, w=x_{13}$. Then the maximum number of internally disjoint S-trees connecting S in H is $\left(n(k-1)-1\right)$, see figure 9.

For the three cases we get $\left\lfloor \frac{(2k-3)n}{2}\right\rfloor \le \kappa(S)\le (n(k-1))$, then we deduce that $\kappa_3(K_k(n))\ge \left\lfloor \frac{(2k-3)n}{2}\right\rfloor$. Thus $\left\lfloor \frac{(2k-3)n}{2}\right\rfloor \le \kappa_3(K_k(n))\le n(k-1)-1$. ∎

Theorem 3.4 : Let $L(K_k(n))$ be the line graph of the complete $k-$partite graph with $k,n\ge 3$, then the generalized 3-connectivity of $L(K_k(n))$ is $2\left((k-1)n-2\right)\le \kappa_3(L(K_k(n))\le 2((k-1)n-2)+1$.

Proof: Let $R = L(K_k(n))$ and $M = \bigcup_{i=1}^{k}\left(\bigcup_{j=1}^{n} L(H_{ij})\right)$, from lemma (3.2) we have $R \cong M$. Since the degree of any vertex in M is $\big((2((k-1)n-2)\big)+2$, then M is $\big(2((k-1)n-2)\big)+2$-regular graph, by the proposition (2.2) we have $\kappa_3(L(M)) \le 2((k-1)n-2)+1$. For completing the proof we just need to show that for any 3-subset $S=\{u,v,w\} \subseteq V(M)$, there exist $2((k-1)n)-2$ internally disjoint Steiner trees connecting S in M. Since $L(H_{1j}) = L(H_{2j}) = ... = L(H_{ij}) = K_{(k-1)n}$, $\forall\, i=1,2,...,k,\ j=1,2,...,n$, $L(H_{1j}) \cong ... \cong L(H_{ij})$, then we have three cases:

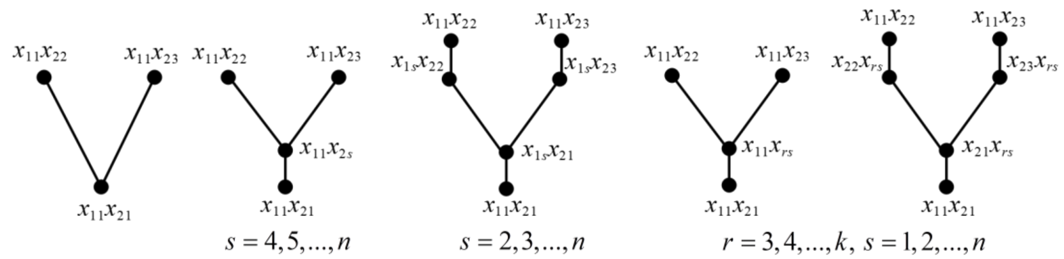

Figure 10

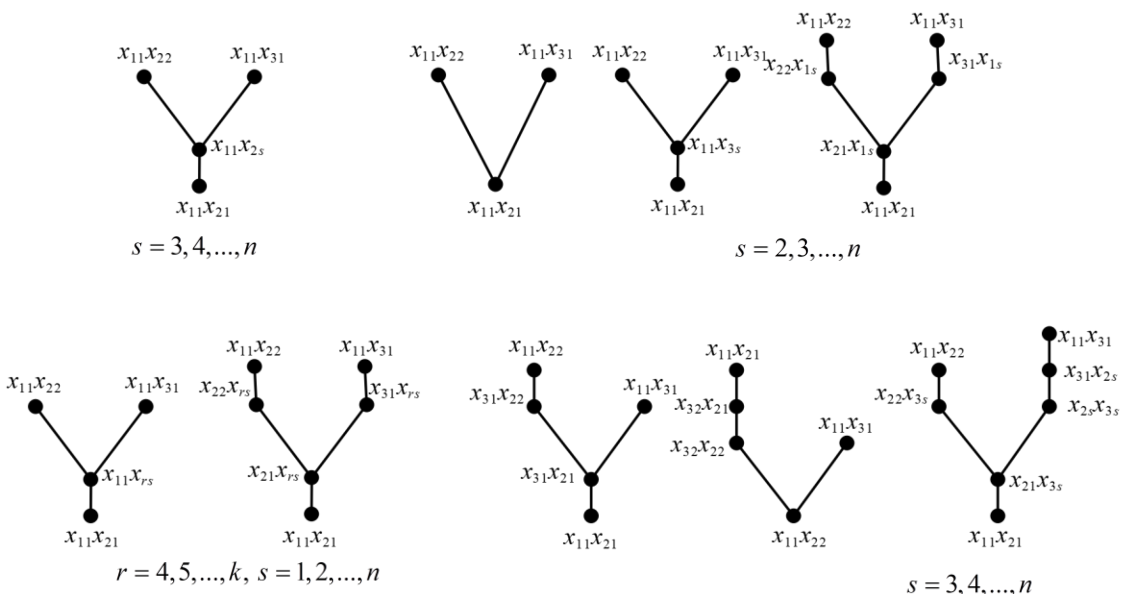

Figure 11

Case 1. If $u,v,w \in L(H_{ij}),\ \forall i=1,2,...,k,\ \ j=1,2,...,n$. Without loss of generality we assume $i=1,\ j=1$, such that $u,v,w \in L(H_{11})$. Then there are two subcases :

Subcase 1.1 Let $u = x_{11}x_{21}$, $v = x_{11}x_{22}$, $w = x_{11}x_{23}$ .Then the maximum number of internally disjoint S-trees of connecting S in M is $\big(2((k-1)n-2)\big)+1$, see figure 10.

Subcase 1.2 Let $u = x_{11}x_{21}$, $v = x_{11}x_{22}$, $w = x_{11}x_{31}$. Then the maximum number of internally disjoint S-trees of connecting S in M is $\big(2((k-1)n-2)\big)+1$, see figure 11.

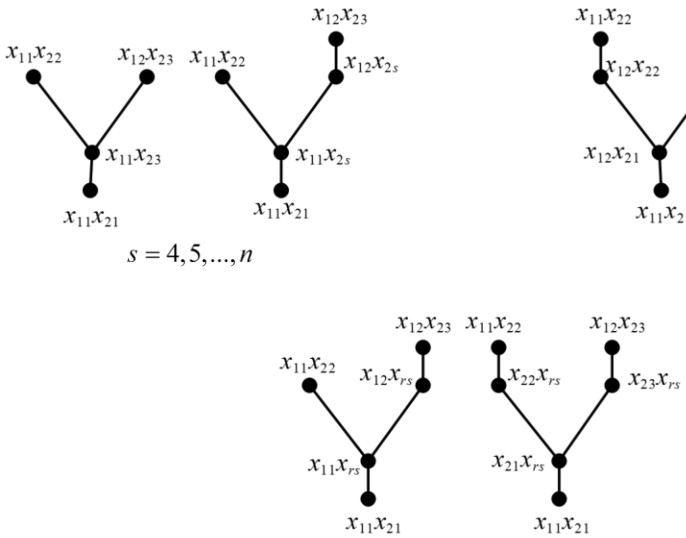

Figure 12

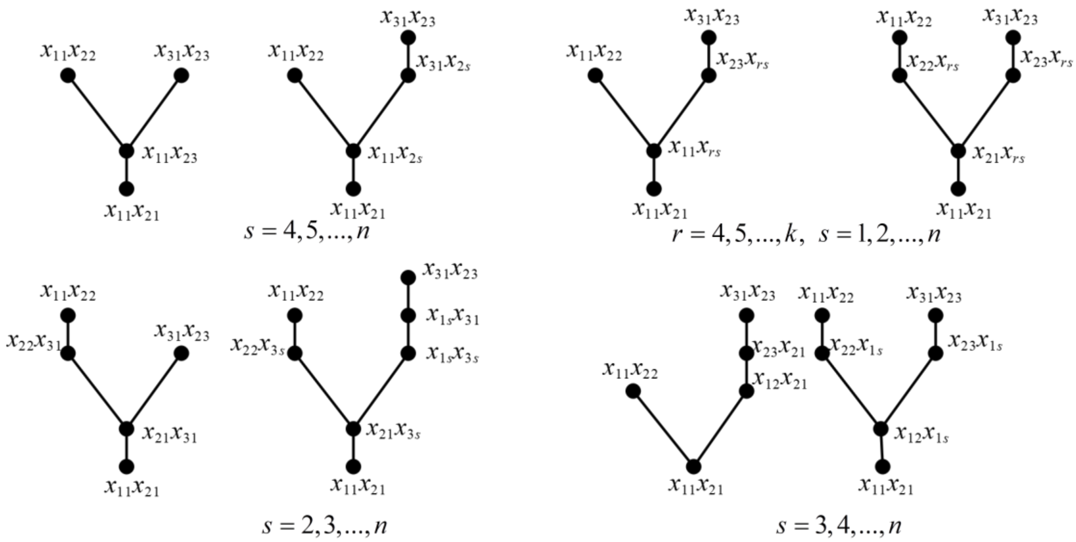

Figure 13

Case 2. If $u,v \in L(H_{ij}),\ w \notin L(H_{ij})$, $\forall i = 1,2,\dots,k,\ j = 1,2,\dots,n$. Again assume $i = 1,\ j = 1$, such that $u,v \in L(H_{11}), w \notin L(H_{11})$. Then there are four subcases :

Subcase 2.1 Let $u = x_{11}x_{21},\ v = x_{11}x_{22},\ w = x_{12}x_{23}$. Then the maximum number of internally disjoint S-trees of connecting S in M is $\left(2((k-1)n-2)\right)+1$, see figure 12.

.**Subcase 2.2** Let $u = x_{11}x_{21}, v = x_{11}x_{22},\ w = x_{31}x_{23}$.Then the maximum number of internally disjoint S-trees of connecting S in M is $\left(2((k-1)n-2)\right)+1$, see figure 13.

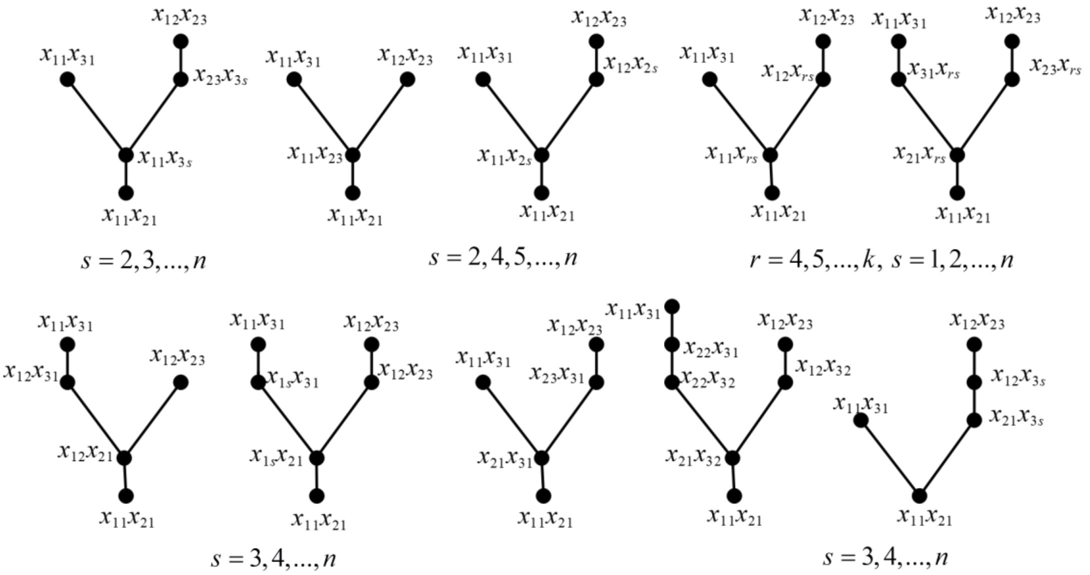

Figure 14

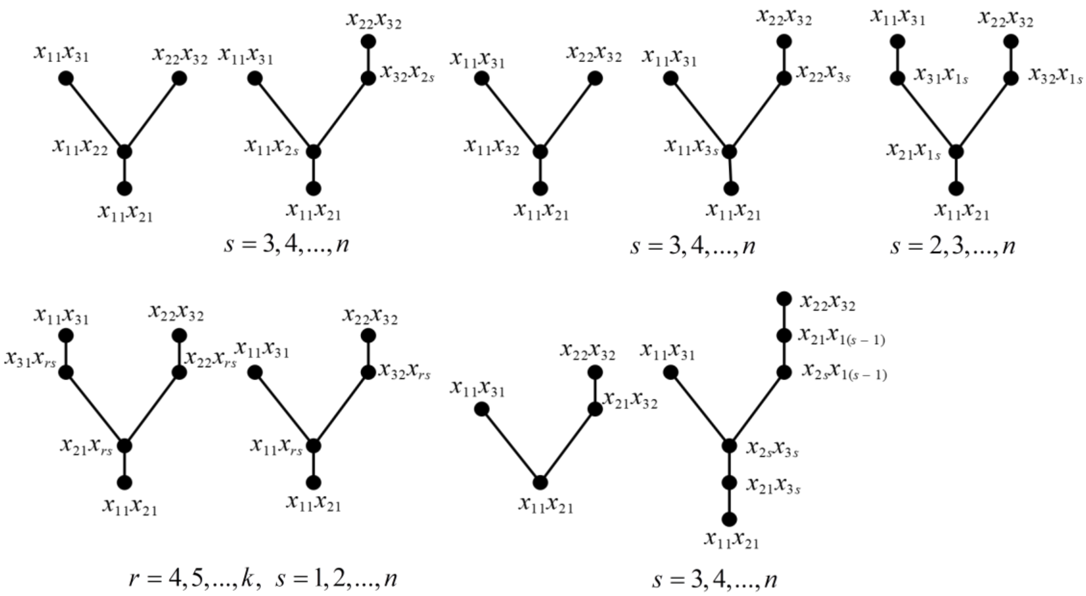

Figure 15

Subcase 2.3 Let $u = x_{11}x_{21},\ v = x_{11}x_{22},\ w = x_{11}x_{23}$.Then the maximum number of internally disjoint S-trees of connecting S in M is $\left(2((k-1)n-2)\right)+1$, see figure 14.

Subcase 2.4 Let $u = x_{11}x_{21}$, $v = x_{11}x_{31}$, $w = x_{22}x_{32}$.Then the maximum number of internally disjoint S-trees of connecting S in M is $\left(2((k-1)n-2)\right)$, see figure 15.

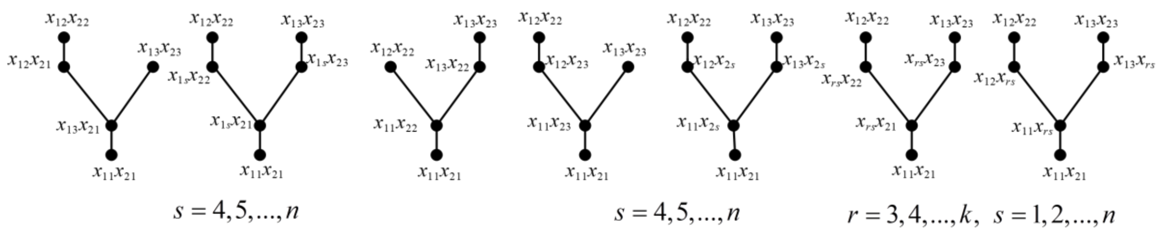

Figure 16

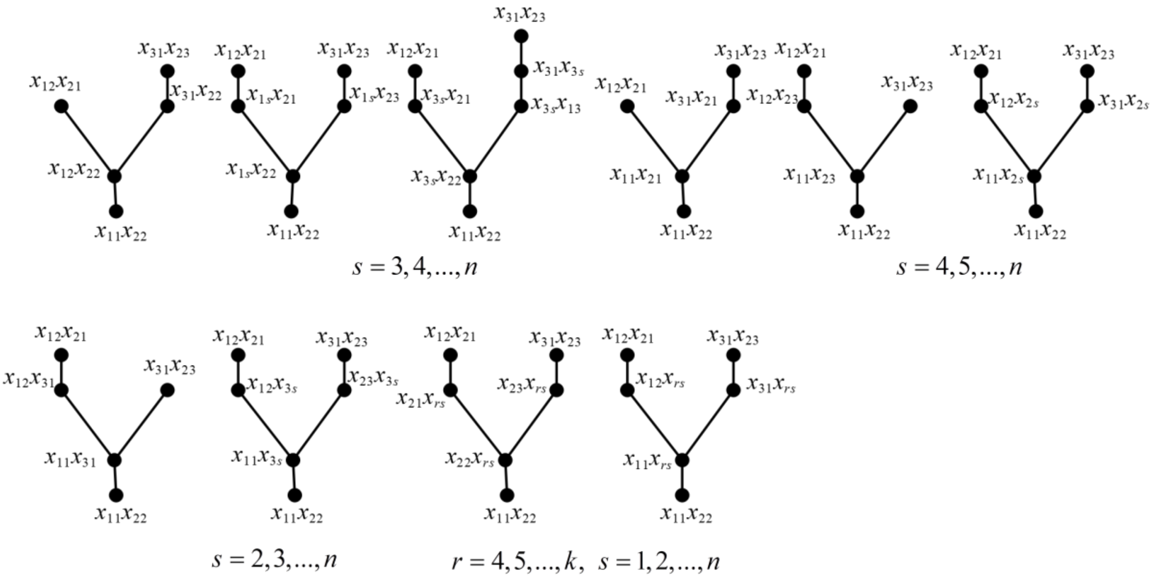

Figure 17

Case 3. If $u\in L(H_{ij}),v,w\notin L(H_{ij}),\forall i=1,2,\ldots,k,\ j=1,2,\ldots,n$. Assume $i=1,\ j=1$ such that $u\in L(H_{ij}),\ v,w\notin L(H_{ij})$. Then there are two subcases:

Subcase 3.1 Let $u=x_{11}x_{21},\ v=x_{12}x_{22},\ w=x_{13}x_{23}$. Then the maximum number of internally disjoint S-trees of connecting S in M is $\left(2((k-1)n-2)\right)+1$, see figure 16.

Subcase 3.2 Let $u=x_{11}x_{22},\ v=x_{21}x_{12},\ w=x_{31}x_{23}$. Then the maximum number of internally disjoint S-trees of connecting S in M is $\left(2((k-1)n-2)\right)$, see figure 17.

From the cases that we discussed we get $2\left((k-1)n-2\right)\le\kappa(S)\le 2\left((k-1)-2\right)+1$. Then $\kappa_3(L(K_k(n))\ge 2\left((k-1)n-2\right)$. Therefore $2\left((k-1)n-2\right)\le\kappa_3(L(K_k(n))\le 2\left((k-1)n-2\right)+1$.■

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أتصال 3- المعمم للبيان الجزئي -k التام المتساوي ولبيانه الخطي

ضياء داخل كاظم , علاء عامر نجم

قسم الرياضيات , كلية العلوم ,جامعة البصرة

المستخلص

للحصول على مجموعة رؤوس V من اصل 2 على الاقل في البيان G فإننا بحاجة الى شجرة من اجل توصيل المجموعة, حيث عادة ما تسمى هذه الشجرة بشجرة ستاينر ربط S (او شجرة S) يقال عن شجرتين من اشجار ستاينر مثل T, T' انهما منفصلتان داخليا اذا كان $E(T) \bigcap E(T') = \phi, V(T) \bigcap V(T') = S$. لتكن $\kappa_G(S)$ تشير الى الحد الاقصى لعدد اشجار ستاينر المنفصلة داخليا والتي تربط S في G. اتصال k المعمم $\kappa(G)$ للبيان G والذي تم تقديمه من قبل الباحث Chartrand (1984) يعرف بانه $\kappa_k(G) = \min\{\kappa_G(S) : S \subseteq V(G)\, and\, |S| = k\}$. في هذا البحث حددنا اتصال 3 المعمم للبيان الجزئي k التام المتساوي وبيانه الخطي.