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On $C_5 \oplus C_{12}$ –Manifolds

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Abstract
In this paper, we explore the geometric properties and tensor components of $C_5 \oplus C_{12} -$ manifolds. Firstly, we established the characteristics identity of this class on G-structure adjoined space and found the equivalent conditions for the defining condition of the class in terms of Kirichenko’s tensors. Furthermore, the Cartan structure equations, components of the Riemannian curvature tensor, and the Ricci tensor are derived of this class. Finally, we introduced the appropriate conditions for these manifolds to be Einstein manifolds.

Keywords: Almost contact metric manifold, Kenmotsu manifolds, Einstein manifold, Cartan’s structure equations, Riemannian curvature tensor.

حول منطويات – $C_5 \oplus C_{12}$

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الخلاصة
في هذا البحث, نستكشف الخصائص الهندسية ومركبات التنسور لمنطويات $C_5 \oplus C_{12}$. في البدء, انشأنا المعادلة المميزة لهذه الفئة على الفضاء المرافق للبنية G ووجدنا الشروط المكافئة للشرط التعريفي للفئة بدلالة مركبات تناسر كيريجينكا. علاوة على ذلك، معادلات كارتان التركيبية ومركبات تناسر الانحناء الريماني وريشي تم اشتقاقها لهذه الفئة. اخيراً، قدمنا الشروط المناسبة التي تجعل هذه المنطويات تكون منطويات اينشتاين.

1. Introduction

One of the important studies on almost contact metric manifolds is the manifolds of a class and this class forms a direct sum of some irreducible classes that constituted by Chinea and Gonzalez [1]. The most common such classes are $C_5 \oplus C_6 -$class, $C_6 \oplus C_7 -$class, $C_2 \oplus C_9 -$class and $C_5 \oplus C_{12} -$class. The classes $C_5 \oplus C_6$ and $C_6 \oplus C_7$ are normal. Whereas, both the classes $C_2 \oplus C_9$ and $C_5 \oplus C_{12}$ are not normal but they are different from each other in an essential part that $C_5 \oplus C_{12} -$class has a proper normal subclass while $C_2 \oplus C_9 -$class has not because the normal manifolds are being of class $C_3 \oplus C_4 \oplus C_5 \oplus C_6 \oplus C_7 \oplus C_8$. The manifolds of classes $C_5 \oplus C_6$, $C_6 \oplus C_7$ and $C_2 \oplus C_9$ are said to be trans – Sasakian manifolds, quasi – Sasakian manifolds and almost cosymplectic manifolds respectively. Whereas, the $C_5 \oplus C_{12} -$class is not bear a famous name.

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Trans – Sasakian manifolds firstly discovered by Oubiña [2] in 1985. Later, Marrero [3] discussed the trans – Sasakian manifolds with local structure and dimension ≥ 5 and furnished his study by examples. Kirichenko and Rodina [4] identified the almost trans – sasakian class and characterized it. They also studied the trans – sasakian class of non- integrable structure with constant Φ- holomorphic sectional curvature. Recently, Rustanov et al. [5] studied the nearly trans – Sasakian manifolds from a linear extension to a special class of almost Hermitian manifolds. Moreover, many articles related to the article in citation [5] were done by Rustanov [6, 7] and Rustanov and Kharitonova [8]. On the other hand, Rahman and Rai [9] introduced a generalized type of submanifolds from nearly trans – Sasakian manifolds.

Quasi – Sasakian manifolds appeared first time by Blair [10] in 1967. Kirichenko and Rustanov [11] studied quasi – Sasakian manifolds on the G – structure adjoined space. Aristarkhova [12] investigated some tensors geometries of quasi – Sasakian manifolds. Cappelletti-Montano et al. [13] discussed the geometry of manifold with 3- contact structures and each structure is quasi- sasakian structure. While Di Pinto and Dileo [14] introduced anti-quasi- sasakian class that its intersection with quasi- sasakian class is the co- K$\ddot{a}$hler manifold. The almost cosymplectic manifolds considered by many authors but the $C_5 \oplus C_{12}$ −class introduced and studied only by Falcitelli [15] and de Candia and Falcitelli in few articles such as [16, 17, 18].

Therefore, our study focused on $C_5 \oplus C_{12}$ −class on G – structure adjoined space and divided into the characterization of $C_5 \oplus C_{12}$ −class in section 3, first and second groups of Cartan's structure equations for $C_5 \oplus C_{12}$ −class in section 4, and Riemannian curvature and Ricci tensors of $C_5 \oplus C_{12}$ −class between theory and application in section 5.

2. Preliminaries

We use the notation M^{2n+1}, g and d to represent a smooth manifold with odd dimension, a Riemannian metric and exterior differentiation operator, respectively. Additionally, $X(M)$ represents the Lie algebra of vector fields over M^{2n+1}.

Definition 2.1 [11]. Let (M^{2n+1}, g) stand for a Riemannian manifold. The triple (ξ, η, Φ) with the foregoing Riemannian manifold, where ξ is a vector field, η is 1-form, and Φ is (1,1)-tensor over $X(M)$, which satisfies the following conditions:

1. $\Phi(\xi) = 0$,
2. $\eta(\xi) = 1$,
3. $\eta \circ \Phi = 0$,
4. $\Phi^2(X) = -X + \eta(X)\xi$,
5. $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$; for all $X, Y \in X(M)$,

is called an almost contact metric (AC-) manifold.

Example 2.2 [19]. Suppose that $\mathbb{R}^{2n+1} = \{(x_1, \dots, x_n, y_1, \dots, y_n, z) : x_i, y_i, z \in \mathbb{R}, \text{ for all } i = \{1, \dots, n\}$. If we take $\xi = 2\frac{\partial}{\partial z}, \eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y_i\, dx_i)$ and $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}((dx_i)^2 + (dy_i)^2)$ and Φ is given by the matrix

$$\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{pmatrix},$$

then $\mathbb{R}^{2n+1}$ with this structure is an AC-manifold.

For any orthonormal basis $\{\xi, e_1, \dots, e_n, e_{\hat{1}}, \dots, e_{\hat{n}}\}$ of $X(M)$, we define an A-frame as $(p; \xi, \varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}})$ where p is any point in M, $\varepsilon_a = \sqrt{2}\sigma(e_a), \varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_a)$, $\sigma = \frac{1}{2}(\mathrm{id} - \sqrt{-1}\Phi)$; $\bar{\sigma} = \frac{1}{2}(\mathrm{id} + \sqrt{-1}\Phi)$, $a = 1, \dots, n$ and $\hat{a} = a + n$. The set of all such frames determines a G-structure on M^{2n+1}, whose structure group is the Lie group $U(n) \times \{e\}$ (see [20, 21, 22]).

These frames are characterized by the fact that the matrices of the tensors g and Φ have the form [23] :

$$\left(g_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & O & I_n \\ 0 & I_n & O \end{pmatrix}; \quad \left(\Phi^i_j\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & O \\ 0 & O & -\sqrt{-1}I_n \end{pmatrix}, \tag{2.1}$$

where O and I_n are zeros matrix and $n \times n$ identity matrix, respectively. Also $i, j = 0,1,\dots,2n$. Kirichenko defined six tensors represented by the following formulas [24]:

$$\begin{aligned} B(X,Y) &= -\frac{1}{8}\{\Phi \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi^2 X) + \Phi \circ \nabla_{\Phi Y}(\Phi)(\Phi X) + \Phi^2 \circ \nabla_{\Phi Y}(\Phi)(\Phi^2 X) \\ &\quad -\Phi^2 \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi X)\}; \\ C(X,Y) &= -\frac{1}{8}\{-\Phi \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi^2 X) + \Phi \circ \nabla_{\Phi Y}(\Phi)(\Phi X) + \Phi^2 \circ \nabla_{\Phi Y}(\Phi)(\Phi^2 X) \\ &\quad +\Phi^2 \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi X)\}; \\ D(X) &= \frac{1}{4}\left\{2\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi - 2\Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi - \Phi \circ \nabla_{\xi}(\Phi)(\Phi^2 X) + \Phi^2 \circ \nabla_{\xi}(\Phi)(\Phi X)\right\}; \\ E(X) &= -\frac{1}{2}\{\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi + \Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi\}; \\ F(X) &= \frac{1}{2}\{\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi - \Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi\}; \\ G &= \Phi \circ \nabla_{\xi}(\Phi)\xi = \nabla_{\xi}\xi. \end{aligned} \tag{2.2}$$

Theorem 2.3 [24]. The components of the above Kirichenko's tensors are all equal to zero except for the components defined by the following formulas, respectively:

1. $B^{ab}{}_c = -\frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},c}$; $\quad B_{ab}{}^c = \frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,\hat{c}}$;
2. $C^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},\hat{c}}$; $\quad C_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,c}$;
3. $B^{ab} = \sqrt{-1}\left(\Phi^a_{0,\hat{b}} - \frac{1}{2}\Phi^a_{\hat{b},0}\right)$; $\quad B_{ab} = -\sqrt{-1}\left(\Phi^{\hat{a}}_{0,b} - \frac{1}{2}\Phi^{\hat{a}}_{b,0}\right)$;
4. $B^a{}_b = \sqrt{-1}\Phi^a_{0,b}$; $\quad B_a{}^b = -\sqrt{-1}\Phi^{\hat{a}}_{0,\hat{b}}$;
5. $F^{ab} = \sqrt{-1}\Phi^0_{\hat{a},\hat{b}}$; $\quad F_{ab} = -\sqrt{-1}\Phi^0_{a,b}$;
6. $C^a = -\sqrt{-1}\Phi^0_{\hat{a},0}$; $\quad C_a = \sqrt{-1}\Phi^0_{a,0}$;

where $\Phi^i_{j,k}$ are the components of $\nabla\Phi$ on G- structure adjoined space, $a, b, c = 1,\dots,n$ and $\hat{a} = a + n$.

Now, suppose that θ is the 1-form of the Riemannian connection ∇ and $\{\omega^0 = \omega, \omega^1,\dots,\omega^{2n}\}$ is the dual A-frame on M. From [24] we have the following:

$$\begin{aligned} &\theta^a_{\hat{b}} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},k}\omega^k; &&\theta^{\hat{a}}_b = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,k}\omega^k; &&\Phi^a_{b,k} = 0; \\ &\theta^0_{\hat{a}} = \sqrt{-1}\Phi^0_{\hat{a},k}\omega^k; &&\theta^0_a = -\sqrt{-1}\Phi^0_{a,k}\omega^k; &&\Phi^{\hat{a}}_{\hat{b},k} = 0; \\ &\theta^{\hat{a}}_0 = -\sqrt{-1}\Phi^{\hat{a}}_{0,k}\omega^k; &&\theta^a_0 = \sqrt{-1}\Phi^a_{0,k}\omega^k; &&\Phi^0_{0,k} = 0. \end{aligned} \tag{2.3}$$

Moreover, $\theta^i_j + \theta^{\hat{j}}_{\hat{i}} = 0$; $\Phi^i_{j,k} = -\Phi^{\hat{j}}_{\hat{i},k}$; $\theta^0_0 = 0$, where $i, j, k = 0, a, \hat{a}$ and $\hat{\hat{i}} = i$.

3. The Characterization of $C_5 \oplus C_{12}$-Manifolds

The defining condition of any AC-manifold which falls in $C_5 \oplus C_{12}$ −class was defined by de Candia and Falcitelli [17] as follows:

$$\nabla_X(\Phi)Y = \alpha\{g(\Phi X, Y)\xi - \eta(Y)\Phi X\} - \eta(X)\{\eta(Y)\Phi(\nabla_\xi \xi) + g(\nabla_\xi \xi, \Phi Y)\xi\}, \qquad (3.1)$$

where α is a known smooth function related to η.

Now, we can write Equation (3.1) on the G-structure adjoined space as follows:

$$\Phi^i_{j,k} = \alpha(g_{mj}\Phi^m_k \delta^i_0 - \eta_j \Phi^i_k) - \eta_k(\eta_j \Phi^i_l G^l + \Omega_{lj} G^l \delta^i_0), \qquad (3.2)$$

where G^l are the components of the sixth Kirichenko's tensor G on G- structure adjoined space, and Ω is a skew symmetric tensor defined by $\Omega(X,Y) = g(X, \Phi Y), \quad \forall\ X, Y \in X(M)$.

Theorem 3.1. Let M^{2n+1} be an AC-manifold, then the following statements are equivalent:

1. M has $C_5 \oplus C_{12} -$ structure.
2. $B = C = D = F = 0; E = -\alpha\ \Phi^2$.
3. On the G-structure adjoined space, we have

$$B^{ab}{}_c = B_{ab}{}^c = C^{abc} = C_{abc} = B^{ab} = B_{ab} = F^{ab} = F_{ab} = 0$$

$B^a{}_b = B_b{}^a = \alpha\delta^a_b$.

Proof. Employing a direct proof approach, demonstrating that each statement implies the other two. Assume the first statement be true, from Definition 2.1 and Equation (3.1) we obtain that $\nabla_{\Phi^2 X}(\Phi)\xi = \alpha\Phi X$, so $\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi = \alpha\Phi^2 X$. Also, $\nabla_{\Phi X}(\Phi)\xi = -\alpha\Phi^2 X$, therefore $\Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi = \alpha\Phi^2 X$. Hence

$E(X) = -\frac{1}{2}\{\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi + \Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi\} = -\frac{1}{2}\{\alpha\Phi^2 X + \alpha\Phi^2 X\} = -\alpha\Phi^2 X$, that means $E = -\alpha\ \Phi^2$, and similarly we get $B = C = D = F = 0$. Now suppose the second statement, by taking $i = a,\ j = 0, k = b$ in Equation (3.2) we obtain $\Phi^a_{0,b} = -\sqrt{-1}\ \alpha\ \delta^a_b$, thus $B^a{}_b = \sqrt{-1}\ \Phi^a_{0,b} = \alpha\delta^a_b$. And clearly for other components.

Corollary 3.2. Let M^{2n+1} be an AC-manifold of $C_5 \oplus C_{12}$ class, then

$$B^{abc} = B_{abc} = C^{ab} = C_{ab} = 0,$$

where $B^{abc} = C^{a[bc]}$, $B_{abc} = C_{a[bc]}$, $C^{ab} = F^{[ab]}$, $C_{ab} = F_{[ab]}$.

Proof. Since $C^{a[bc]} = \frac{1}{2}\{ C^{abc} - C^{acb} \}$ and so for the other components, then the results happened from Theorem 3.1; item 3. ■

Theorem 3.3. The components of $\nabla\Phi$ on G- structure adjoined space of $C_5 \oplus C_{12}$ class has the following values:

$$\Phi^a_{\hat{b},k} = \Phi^a_{0,\hat{c}} = \Phi^0_{b,c} = 0; \quad \Phi^a_{0,c} = -\sqrt{-1}\ \alpha\ \delta^a_c; \quad \Phi^0_{b,\hat{c}} = -\sqrt{-1}\ \alpha\ \delta^c_b\ .$$

Proof. By taking $i = a, j = 0, k = c$ in Equation (3.2) and taking into account Equation (2.1), we obtain $\Phi^a_{0,c} = -\sqrt{-1}\ \alpha\ \delta^a_c$, and similarly for others components. ■

Theorem 3.4. Let M^{2n+1} be an AC-manifold of $C_5 \oplus C_{12}$ class, then we have:

$$\theta^a_{\hat{b}} = 0; \quad \theta^{\hat{a}}_b = \overline{\theta^a_{\hat{b}}}; \quad \theta^0_0 = 0;$$

$\theta^0_{\hat{a}} = -C^a\omega - \alpha\omega^a; \quad \theta^0_a = \overline{\theta^0_{\hat{a}}}$,

where $\overline{\omega_a} = \omega^a$ and $\overline{\omega^a} = \omega_a$.

Proof. According to Equation (2.3), Theorem 2.3, Theorem 3.1, and taking into account Theorem 3.3, we have

$\theta^a_{\hat{b}} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},k}\omega^k = 0$, (from Theorem 3.3)

$$\theta^0_{\hat{a}} = \sqrt{-1}\Phi^0_{\hat{a},k}\omega^k = \sqrt{-1}\Phi^0_{\hat{a},0}\omega^0 + \sqrt{-1}\Phi^0_{\hat{a},c}\omega^c + \sqrt{-1}\Phi^0_{\hat{a},\hat{c}}\omega^{\hat{c}} = -C^a\omega - \alpha\omega^a.$$

4. Cartan's Structure Equations of $C_5 \oplus C_{12}$-Manifolds

In this section, we calculated the structure equations of $C_5 \oplus C_{12}$-manifolds.

Lemma 4.1 [24, 25]. Let M^{2n+1} be an AC-manifold, then the first family of structure equations given by:

1. $d\omega^a = -\theta_b^a \wedge \omega^b + B^{ab}{}_c\, \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B^a{}_b\, \omega \wedge \omega^b + B^{ab}\omega \wedge \omega_b$;
2. $d\omega_a = \theta_a^b \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_{abc}\, \omega^b \wedge \omega^c + B_a{}^b\omega \wedge \omega_b + B_{ab}\, \omega \wedge \omega^b$;
3. $d\omega = C_{bc}\, \omega^b \wedge \omega^c + C^{bc}\omega_b \wedge \omega_c + C_c^b\omega^c \wedge \omega_b + C_b\, \omega \wedge \omega^b + C^b\omega \wedge \omega_b$,

where $C_c^b = B^b{}_c - B_c{}^b$.

In the following theorem, we found the first family of structure equations of $C_5 \oplus C_{12}$-manifolds.

Corollary 4.2. Let M^{2n+1} be an AC-manifold of $C_5 \oplus C_{12}$ class, then the first group of structure equations given in the following forms:

1. $d\omega^a = -\theta_b^a \wedge \omega^b + \alpha\omega \wedge \omega^a$;
2. $d\omega_a = \theta_a^b \wedge \omega_b + \alpha\omega \wedge \omega_a$;
3. $d\omega = C_b\omega \wedge \omega^b + C^b\omega \wedge \omega_b$.

Proof. The result directly follows from Lemma 4.1 and Theorem 3.1; item 3. ■

Theorem 4.3. Let M^{2n+1} be an AC-manifold of $C_5 \oplus C_{12}$ class, then the second group of structure equations given in the following form:

1. $dC_b = C_h\theta_b^h + C_{bh}\omega^h + C_b{}^h\omega_h + C_{b0}\omega$;
2. $dC^b = -C^h\theta_h^b + C^{bh}\omega_h + C^b{}_h\omega^h + C^{b0}\omega$;
3. $d\theta_b^a = -\theta_h^a \wedge \theta_b^h + A_{bh}^{ad}\omega^h \wedge \omega_d + A_{bh0}^a\omega^h \wedge \omega + A_b^{ah0}\omega_h \wedge \omega$;
4. $d\alpha = \alpha_d\omega^d + \alpha^d\omega_d + \alpha_0\omega$,

where $h = 1, \dots, n,$ $A_{[bh]}^{ad} = A_{bh}^{[ad]} = A_{[bh]0}^a = A_b^{[ah]0} = C_{[bh]} = C^{[bh]} = 0, C_b{}^h = C^h{}_b$ and for $n > 1$, we have $\alpha^d = \alpha C^d$ and $\alpha_d = \alpha C_d$.

Proof. Acting the exterior derivative d on Corollary 4.2; item 3, we get:

$$0 = dC_b \wedge \omega \wedge \omega^b + C_b(C_a\omega \wedge \omega^a + C^a\omega \wedge \omega_a) \wedge \omega^b - C_b\omega \wedge \left(-\theta_h^b \wedge \omega^h + \alpha\omega \wedge \omega^b\right)$$
$$+dC^b \wedge \omega \wedge \omega_b + C^b(C_a\omega \wedge \omega^a + C^a\omega \wedge \omega_a) \wedge \omega_b - C^b\omega \wedge \left(\theta_b^h \wedge \omega_h + \alpha\omega \wedge \omega_b\right).$$

After changing some indexes of the above equation, we obtain:

$$\left(dC_b - C_h\theta_b^h\right) \wedge \omega \wedge \omega^b \quad +C_{[b}C_{a]}\omega \wedge \omega^a \wedge \omega^b + \left(dC^b + C^h\theta_h^b\right) \wedge \omega \wedge \omega_b$$
$$+C^{[b}C^{a]}\omega \wedge \omega_a \wedge \omega_b = 0.$$

Since $C_{[b}C_{a]} = \frac{1}{2}(C_bC_a - C_aC_b) = 0$ and $C^{[b}C^{a]} = \frac{1}{2}(C^bC^a - C^aC^b) = 0$, then the above equation reduced to:

$$\left(dC_b - C_h\theta_b^h\right) \wedge \omega \wedge \omega^b + \left(dC^b + C^h\theta_h^b\right) \wedge \omega \wedge \omega_b = 0. \tag{4.1}$$

Since $\left(dC_b - C_h\theta_b^h\right)$ and $\left(dC^b + C^h\theta_h^b\right)$ are 1-forms, then they can be written by the following formulae:

$$dC_b - C_h\theta_b^h = C_{bd}^h\theta_h^d + C_{bh}\omega^h + C_b{}^h\omega_h + C_{b0}\omega,$$
$$dC^b + C^h\theta_h^b = C_h^{bd}\theta_d^h + C^{bh}\omega_h + C^b{}_h\omega^h + C^{b0}\omega.$$

Then by substitution the above formulae in Equation (4.1), we get $C_{bd}^h = C_h^{bd} = C_{[bh]} = C^{[bh]} = 0$ and $C_b{}^h = C^h{}_b$.

Now, by the same way above on Corollary 4.2; item 1, then we get:

$$-\left(d\theta_b^a + \theta_h^a \wedge \theta_b^h\right) \wedge \omega^b + \alpha C_b\omega^a \wedge \omega \wedge \omega^b + \alpha C^b\omega^a \wedge \omega \wedge \omega_b - d\alpha \wedge \omega^a \wedge \omega = 0. \tag{4.2}$$

Since $d\theta_b^a + \theta_h^a \wedge \theta_b^h$ and $d\alpha$ are 2-form and 1-form respectively, then they can written by the family of basis on G-structure adjoined space as follows:

$$d\theta_b^a + \theta_h^a \wedge \theta_b^h = A_{bhf}^{adc}\theta_d^h \wedge \theta_c^f + A_{bhc}^{ad}\theta_d^h \wedge \omega^c + A_{bh}^{adc}\theta_d^h \wedge \omega_c + A_{bh0}^{ad}\theta_d^h \wedge \omega + A_{bhd}^{a}\omega^h \wedge \omega^d$$
$$+A_{bh}^{ad}\omega^h \wedge \omega_d + A_{bh0}^{a}\omega^h \wedge \omega + A_{b}^{ahd}\omega_h \wedge \omega_d + A_{b}^{ah0}\omega_h \wedge \omega.$$
$$d\alpha = \alpha_d\omega^d + \alpha^d\omega_d + \alpha_0\omega.$$

Then Equation (4.2) becomes:

$$-A_{bhf}^{adc}\theta_d^h \wedge \theta_c^f \wedge \omega^b - A_{[b|h|c]}^{ad}\theta_d^h \wedge \omega^c \wedge \omega^b - A_{bh}^{adc}\theta_d^h \wedge \omega_c \wedge \omega^b - A_{bh0}^{ad}\theta_d^h \wedge \omega \wedge \omega^b - A_{[bhd]}^{a}\omega^h \wedge \omega^d \wedge \omega^b - A_{[bh]}^{ad}\omega^h \wedge \omega_d \wedge \omega^b - A_{[bh]0}^{a}\omega^h \wedge \omega \wedge \omega^b - A_{b}^{ahd}\omega_h \wedge \omega_d \wedge \omega^b - A_{b}^{ah0}\omega_h \wedge \omega \wedge \omega^b + \alpha C_b\omega^a \wedge \omega \wedge \omega^b + \alpha C^b\omega^a \wedge \omega \wedge \omega_b - \alpha_d\omega^d \wedge \omega^a \wedge \omega - \alpha^d\omega_d \wedge \omega^a \wedge \omega = 0.$$

$$-A_{bhf}^{adc}\theta_d^h \wedge \theta_c^f \wedge \omega^b - A_{[b|h|c]}^{ad}\theta_d^h \wedge \omega^c \wedge \omega^b - A_{bh}^{adc}\theta_d^h \wedge \omega_c \wedge \omega^b - A_{bh0}^{ad}\theta_d^h \wedge \omega \wedge \omega^b$$
$$- A_{[bhd]}^{a}\omega^h \wedge \omega^d \wedge \omega^b - A_{[bh]}^{ad}\omega^h \wedge \omega_d \wedge \omega^b - A_{b}^{ahd}\omega_h \wedge \omega_d \wedge \omega^b$$
$$- A_{[bh]0}^{a}\omega^h \wedge \omega \wedge \omega^b + \alpha C_{[b}\delta_{h]}^a\omega^h \wedge \omega \wedge \omega^b + \alpha_{[h}\delta_{b]}^a\omega^h \wedge \omega \wedge \omega^b$$
$$- A_{b}^{ah0}\omega_h \wedge \omega \wedge \omega^b - \alpha\delta_b^a C^b\omega_h \wedge \omega \wedge \omega^b + \alpha^h\delta_b^a\omega_h \wedge \omega \wedge \omega^b = 0.$$

So, we get:

$$\begin{aligned}
&1)\quad A_{bhf}^{adc} = A_{[b|h|c]}^{ad} = A_{bh}^{adc} = 0;\\
&2)\quad A_{[bhd]}^{a} = A_{[bh]}^{ad} = A_{b}^{ahd} = 0;\\
&3)\quad A_{bh0}^{ad} = 0;\\
&4)\quad A_{[bh]0}^{a} - \alpha C_{[b}\delta_{h]}^a - \alpha_{[h}\delta_{b]}^a = 0;\\
&5)\quad A_{b}^{ah0} - \alpha^h\delta_b^a + \alpha\delta_b^a C^h = 0.
\end{aligned} \tag{4.3}$$

Similarly, Corollary 4.2; item 2, give us:

$$\left(d\theta_a^b - \theta_a^h \wedge \theta_h^b\right) \wedge \omega_b + \alpha C_b\omega_a \wedge \omega \wedge \omega^b + \alpha C^b\omega_a \wedge \omega \wedge \omega_b - d\alpha \wedge \omega_a \wedge \omega = 0. \tag{4.4}$$

According to Equations (4.3), we get:

$$d\theta_a^b - \theta_a^h \wedge \theta_h^b = A_{ahd}^{bc}\theta_c^h \wedge \omega^d + A_{ahc}^{b}\omega^h \wedge \omega^c + A_{ah}^{bc}\omega^h \wedge \omega_c$$
$$+A_{ah0}^{b}\omega^h \wedge \omega + A_{a}^{bh0}\omega_h \wedge \omega.$$

So, Equation (4.4) turns into:

$$A_{ahd}^{bc}\theta_c^h \wedge \omega^d \wedge \omega_b + A_{ahc}^{b}\omega^h \wedge \omega^c \wedge \omega_b + A_{ah}^{[bc]}\omega^h \wedge \omega_c \wedge \omega_b + A_{ah0}^{b}\omega^h \wedge \omega \wedge \omega_b$$
$$+A_{a}^{[bh]0}\omega_h \wedge \omega \wedge \omega_b + \alpha C_b\omega_a \wedge \omega \wedge \omega^b + \alpha C^b\omega_a \wedge \omega \wedge \omega_b - \alpha_h\omega^h \wedge \omega_a \wedge \omega$$
$$-\alpha^h\omega_h \wedge \omega_a \wedge \omega = 0.$$

From above, we have:

$$\begin{aligned}
&1)\quad A_{ahd}^{bc} = A_{ahc}^{b} = A_{bh}^{[ad]} = 0;\\
&2)\quad A_{ah0}^{b} - \delta_a^b\alpha C_h + \delta_a^b\alpha_h = 0;\\
&3)\quad A_{a}^{[bh]0} + \alpha C^{[b}\delta_a^{h]} + \alpha^{[h}\delta_a^{b]} = 0.
\end{aligned} \tag{4.5}$$

Equations (4.3); item 5 and (4.5); item 3 can be rewritten respectively as follow:

$$A_{b}^{[ah]0} + \alpha C^{[h}\delta_b^{a]} - \alpha^{[h}\delta_b^{a]} = 0,$$
$$A_{b}^{[ah]0} - \alpha C^{[h}\delta_b^{a]} + \alpha^{[h}\delta_b^{a]} = 0.$$

Then we get $A_{b}^{[ah]0} = 0$. So $\alpha C^{[h}\delta_b^{a]} - \alpha^{[h}\delta_b^{a]} = 0$. By contracting a and b, we obtain $(n-1)(\alpha\, C^h - \alpha^h) = 0$, and this gives the result. The same can be done for Equations (4.3); item 4 and (4.5); item 2.

5. Riemannian Curvature and Ricci Tensors of $C_5 \oplus C_{12}$-Manifolds

Theorem 5.1. The Riemannian curvature tensor have the following components for the class $C_5 \oplus C_{12}$ on G-structure adjoined space given by:

1. $R^a_{0h0} = C^a{}_h - C^a C_h - \delta^a_h \alpha_0 - \delta^a_h \alpha^2$;
2. $R^a_{0h\hat{d}} = -\delta^a_h \alpha^d$;
3. $R^a_{0\hat{h}0} = C^{ah} - C^a C^h$;
4. $R^a_{0hd} = -2\delta^a_{[h}\alpha_{d]}$;
5. $R^a_{bh0} = A^a_{bh0} - \alpha\delta^a_h C_b$;
6. $R^a_{b\hat{h}0} = A^{ah0}_b + \alpha\delta^h_b C^a$;
7. $R^a_{bh\hat{d}} = A^{ad}_{bh} - \alpha^2\delta^a_h\delta^d_b$;
8. $R^a_{\hat{b}hd} = -2\alpha^2\delta^a_{[h}\delta^b_{d]}$;
9. $R^a_{\hat{b}h0} = 2\alpha C^{[a}\delta^{b]}_h$.

and all others components are zeros or we can find them by the symmetric property or conjugate of above components.

Proof. From [26], any AC-manifold M^{2n+1} satisfies the following 2nd group of Cartan's structure equations:

$$d\theta^i_j + \theta^i_k \wedge \theta^k_j = \frac{1}{2} R^i_{jkl}\omega^k \wedge \omega^l,$$

where R^i_{jkl} denotes to the components of Riemannian curvature tensor R of M^{2n+1} on G-structure adjoined space.

Take $k = 0, h, \hat{h}$ and $l = 0, d, \hat{d}$ in the above equations, we get:

$$d\theta^i_j + \theta^i_0 \wedge \theta^0_j + \theta^i_h \wedge \theta^h_j + \theta^i_{\hat{h}} \wedge \theta^{\hat{h}}_j = R^i_{jh0}\omega^h \wedge \omega + R^i_{j\hat{h}0}\omega_h \wedge \omega + \tfrac{1}{2} R^i_{jhd}\omega^h \wedge \omega^d + R^i_{jh\hat{d}}\omega^h \wedge \omega_d + \tfrac{1}{2} R^i_{j\hat{h}\hat{d}}\omega_h \wedge \omega_d. \tag{5.1}$$

Take several cases depend on the values of $i = 0, a, \hat{a}$; $j = 0, b, \hat{b}$. From Corollary 4.2, Theorem 4.3 and Theorem 3.4, these cases give us the following:

(1) If $i = j = 0$, then the Equation (5.1) give:

$$R^0_{0h0} = R^0_{0\hat{h}0} = R^0_{0hd} = R^0_{0h\hat{d}} = R^0_{0\hat{h}\hat{d}} = 0.$$

(2) If $i = a, j = 0$, then from Equation (5.1), we get:

$$R^a_{0h0} = C^a{}_h - C^a C_h - \delta^a_h \alpha_0 - \delta^a_h \alpha^2; \quad R^a_{0\hat{h}0} = C^{ah} - C^a C^h;$$
$$R^a_{0hd} = -2\delta^a_{[h}\alpha_{d]}; \quad R^a_{0h\hat{d}} = -\delta^a_h \alpha^d; \quad R^a_{0\hat{h}\hat{d}} = 0.$$

(3) If $i = a, j = b$, then from Equation (5.1), we get

$$R^a_{bh0} = A^a_{bh0} - \alpha\delta^a_h C_b; \quad R^a_{b\hat{h}0} = A^{ah0}_b + \alpha\delta^h_b C^a;$$
$$R^a_{bh\hat{d}} = A^{ad}_{bh} - \alpha^2\delta^a_h\delta^d_b; \quad R^a_{bhd} = R^a_{b\hat{h}\hat{d}} = 0.$$

(4) If $i = a, j = \hat{b}$, then according to Equation (5.1) we get

$$R^a_{\hat{b}hd} = -2\alpha^2\delta^a_{[h}\delta^b_{d]}; \quad R^a_{\hat{b}h0} = 2\alpha C^{[a}\delta^{b]}_h:$$
$$R^a_{\hat{b}\hat{h}0} = R^a_{\hat{b}h\hat{d}} = R^a_{\hat{b}\hat{h}\hat{d}} = 0. \qquad \blacksquare$$

Theorem 5.2. The components of Ricci tensor s for the class $C_5 \oplus C_{12}$ on G-structure adjoined space given by:

1. $s_{00} = 2(C^a{}_a - C^a C_a) - 2n(\alpha_0 + \alpha^2)$;
2. $s_{a0} = A^b_{ab0} - n\alpha C_a - (n-1)\alpha_a$;
3. $s_{ab} = C_{ba} - C_b C_a$;
4. $s_{\hat{a}b} = C^a{}_b - C^a C_b - \alpha_0\delta^a_b + A^{ac}_{cb} - 2n\alpha^2\delta^a_b$;

and others components we can find them by the symmetric property or conjugate of above components.

Proof. According to Theorem 5.1 and the following relation (see [7]), we get:

$$\begin{aligned}
s_{ij} &= -R^{k}_{ijk};\\
&= -R^{0}_{ij0} - R^{c}_{ijc} - R^{\hat{c}}_{ij\hat{c}}\,;\ \ i,j,k = 0,1,\dots,2n\\
s_{00} &= -R^{0}_{000} - R^{a}_{00a} - R^{\hat{a}}_{00\hat{a}}\\
&= 2(C^{a}{}_{a} - C^{a}C_{a}) - 2n(\alpha_0 + \alpha^2).\\
s_{a0} &= -R^{0}_{a00} - R^{b}_{a0b} - R^{\hat{b}}_{a0\hat{b}}\\
&= A^{b}_{ab0} - n\alpha C_a - (n-1)\alpha_a.\\
s_{ab} &= -R^{0}_{ab0} - R^{c}_{abc} - R^{\hat{c}}_{ab\hat{c}}\\
&= C_{ba} - C_b C_a.\\
s_{\hat{a}b} &= -R^{0}_{\hat{a}b0} - R^{c}_{\hat{a}bc} - R^{\hat{c}}_{\hat{a}b\hat{c}}\\
&= C^{a}{}_{b} - C^{a}C_b - \alpha_0\delta^{a}_{b} + A^{ac}_{cb} - 2n\alpha^2\delta^{a}_{b}.
\end{aligned}$$

■

Corollary 5.3. The scalar curvature of $C_5 \oplus C_{12}$-manifold on G- structure adjoined space is

$$\kappa = 2A^{ac}_{ca} + 4(C^{a}{}_{a} - C^{a}\,C_a) - 2n(2n+1)\,\alpha^2 - 4n\,\alpha_0.$$

Proof. The scalar curvature κ is defined by $\kappa = g^{ij}\,s_{ij}$, where g^{ij} are the components of the contravariant metric tensor [11].

$$\begin{aligned}
\kappa &= g^{ij}\,s_{ij} = g^{00}\,s_{00} + g^{\hat{a}b}\,s_{\hat{a}b} + g^{b\hat{a}}\,s_{b\hat{a}} = s_{00} + 2\,s_{a\hat{a}}\\
&= 2A^{ac}_{ca} + 4(C^{a}{}_{a} - C^{a}\,C_a) - 4n^2\,\alpha^2 - 2n\,\alpha^2 - 4n\,\alpha_0. \quad \text{(From Theorem 5.2)}
\end{aligned}$$ ■

Definition 5.4. [27] The AC-manifold is called an η-Einstein manifold if

$$s(X,Y) = \lambda\ g(X,Y) + \mu\ \eta(X)\eta(Y);\quad \text{for all } X,Y \in X(M),$$

where λ and μ are scalar functions. If $\mu = 0$ then M is called an Einstein manifold.

Theorem 5.5. The manifold of $C_5 \oplus C_{12}$ class is an Einstein manifold if and only if the following conditions hold:

$$2(C^{a}{}_{a} - C^{a}C_a) - 2n(\alpha_0 + \alpha^2) = \lambda;\quad C_{ba} = C_b C_a;\quad A^{b}_{ab0} = (2n-1)\alpha C_a;$$
$$A^{ac}_{ca} = (2n-1)\{\,C^{a}{}_{a} - C^{a}C_a - n\alpha_0\,\}.$$

Proof. Equation (2.1) and Definition 5.4, give the following:
$s_{00} = \lambda g_{00} = \lambda,\quad s_{a0} = s_{ab} = 0$ and $s_{\hat{a}b} = \lambda g_{\hat{a}b} = \lambda\delta^{a}_{b}$.
According to the above and Theorem 5.2, we get:

1) $2(C^{a}{}_{a} - C^{a}C_a) - 2n(\alpha_0 + \alpha^2) = \lambda,$
2) $C_{ba} = C_b C_a,\qquad A^{b}_{ab0} - n\alpha C_a - (n-1)\alpha_a = 0,$
3) $C^{a}{}_{b} - C^{a}C_b - \alpha_0\delta^{a}_{b} + A^{ac}_{cb} - 2n\alpha^2\delta^{a}_{b} = \lambda\delta^{a}_{b}.$

Since $\alpha_a = \alpha C_a$ for $n > 1$, then we get $A^{b}_{ab0} = (2n-1)\alpha C_a$ and this equality also holds if $n = 1$.

Now, by contracting a and b in equality 3) above and adding the result with equality 1), we get:

$$A^{ac}_{ca} = (2n-1)\{ C^{a}{}_{a} - C^{a}C_{a} - n\alpha_0 \},$$

and this gives the result. ■

Remark 5.6. The scalar curvature of an Einstein $C_5 \oplus C_{12}$-manifolds is $\kappa = (2n+1)\lambda$.

Proof. Regarding Corollary 5.3 and Theorem 5.5, we obtain

$$\kappa = 2(C^{a}{}_{a} - C^{a}C_{a}) - 2n(\alpha_0 + \alpha^2) + 2\{ C^{a}{}_{a} - C^{a}C_{a} - n\alpha_0 + A^{ac}_{ca} - 2n^2\alpha^2 \};$$
$$= \lambda + 2n\lambda = (2n+1)\lambda.$$ ■

Theorem 5.7. The manifold of $C_5 \oplus C_{12}$ class is an η-Einstein manifold if and only if the following conditions hold:

$$\lambda = \frac{1}{n}(C^{a}{}_{a} - C^{a}C_{a} + A^{ac}_{ca}) - \alpha_0 - 2n\alpha^2;$$
$$\mu = 2(C^{a}{}_{a} - C^{a}C_{a}) - \frac{1}{n}(C^{a}{}_{a} - C^{a}C_{a} + A^{ac}_{ca}) + (1-2n)\,\alpha_0;$$
$$C_{ba} = C_b C_a; \quad A^{b}_{abo} = (2n-1)\alpha C_a.$$

Proof. Definition 5.4 implies that $s_{00} = \lambda + \mu$; $s_{a0} = s_{ab} = 0$ and $s_{\hat{a}\mathrm{b}} = \lambda\delta^a_b$. Thus
1) $2(C^{a}{}_{a} - C^{a}C_{a}) - 2n(\alpha_0 + \alpha^2) = \lambda + \mu,$
2) $C^{a}{}_{b} - C^{a}C_{b} - \alpha_0\delta^a_b + A^{ac}_{cb} - 2n\alpha^2\delta^a_b = \lambda\delta^a_b.$
The contracting of Equation 2) gives the following:
3) $C^{a}{}_{a} - C^{a}C_{a} - n\alpha_0 + A^{ac}_{ca} - 2n^2\alpha^2 = n\,\lambda.$
So, solving Equations (1) and (3) with respect to μ and λ given the values of them. The other results similar to the way of the proof of Theorem 5.5. ■

Remark 5.8. If λ and μ are constants in Theorem 5.7, then $dA^{ac}_{ca} = 2n(2n-1)\alpha\, d\alpha$.

Proof. Suppose that λ and μ are constants in Theorem 5.7, then the exterior differentiation for the values of μ and λ in the above theorem implies that respectively:

$$\frac{2n-1}{n} d(C^{a}{}_{a} - C^{a}C_{a}) - (2n-1)d\alpha_0 - \frac{1}{n} dA^{ac}_{c\alpha} = 0\,;$$
$$\frac{1}{n} d(C^{a}{}_{a} - C^{a}C_{\alpha}) + \frac{1}{n} dA^{ac}_{c\alpha} - d\alpha_0 - 4n\alpha\, d\alpha = 0\,.$$

So, the solution of above two equations with respect to $dA^{ac}_{c\alpha}$ gives the result.

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