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RESEARCH ARTICLE

Boundary Value Problems Associated with Morse-Novikov Cohomology Groups of Riemannian Manifolds with Boundary

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ABSTRACT

In this paper, we provide an affirmative response to the following question: if Morse-Novikov cohomology groups of M with non-empty boundary ∂M do not vanish, then what topological or analytical (geometrical) constraints can be enforced to ensure that the equation $\mathsf{d}_\theta\omega = \eta$ is solvable for any non-trivial prescribed $[\eta]$ in absolute $H^k_\theta(M)$ or relative $H^k_\theta(M, \partial M)$ Morse-Novikov cohomology groups?. Where $\mathsf{d}_\theta\omega = \mathsf{d}\omega + \theta \wedge \omega$ and $0 \neq [\theta] \in H^1_{dR}(M)$ for any $\omega \in \Omega^k(M)$. Moreover, this motivates us to investigate the integrability requirements for a variety of perturbed Dirichlet problems for d_θ, Neumann problems for δ_θ, and perturbed mixed boundary value problems for the Poisson equation from topological and analytical perspectives. Furthermore, we investigate the analytical properties of the eigenvalues of the spacial kind of Poisson equations and show that they are positive and their corresponding eigenfunctions are L^2-orthogonal. Consequently, this proves that the set of the corresponding eigenfunctions spans a subspace of the orthogonal complement of the kernel of this equation.

Keywords: Boundary value problems of differential forms, Morse-Novikov cohomology, Lichnerowicz cohomology
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1. Introduction

Let M be a closed, compact, oriented, smooth Riemannian manifold of dimension n, and consider $0 \neq [\theta] \in H^1_{dR}(M)$. Let $\Omega^k = \Omega^k(M)$ be the space of smooth differential k-forms on M and $\mathsf{d}_\theta : \Omega^k \longrightarrow \Omega^{k+1}$ the Morse-Novikov coboundary operator perturbed by θ, which is given by $\mathsf{d}_\theta\alpha = \mathsf{d}\alpha + \theta \wedge \alpha$ for each $\alpha \in \Omega^k$. Then we have $\mathsf{d}_\theta^2 = 0$ because of the closedness of θ, and this gives a complex $(\Omega^k(M), \mathsf{d}_\theta)$. As a result, this complex generates the *Morse-Novikov cohomology* group $H^k_\theta(M) = H^k(\Omega^*(M), \mathsf{d}_\theta)$ [1]. However, if θ_1 and θ_2 are cohomologous to θ then $H^k_{\theta_1}(M) \simeq H^k_{\theta_2}(M)$. Hence, Morse-Novikov cohomology is much more difficult to calculate than de Rham cohomology since it depends on θ. A. Lichnerowicz researched Morse-Novikov cohomology initially in [2] to study poisson geometry. Novikov [3] investigated Morse-Novikov cohomology within the framework of Hamiltonian mechanics, and Guedira and Lichnerowicz separately developed it [4]. This cohomology is extremely important in investigating the locally conformally symplectic and locally conformally Kählerian structures [5]. A significant result in this connection was demonstrated by Chen in [6], who demonstrated that the Morse-Novikov cohomology groups are trivial on any Riemannian closed manifold M with almost nonnegative sectional curvature. The fact is that a closed Riemannian manifold with almost nonnegative sectional curvature is known to be an almost nilpotent space [7]. So, Chen's main Theorem 1.1, is actually a consequence of the following topological results:

Theorem 1.1 ([6]): *If M is an almost nilpotent closed Riemannian manifold, then the Morse-Novikov*

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cohomology group $H^k_\theta(M) = 0,\ \forall k \geq 0$ *and* $\forall[\theta] \in H^1_{dR}(M),\ [\theta] \neq 0.$

So, the importance of Morse-Novikov cohomology groups is without question. Actually, all these results and more consider $\partial M = \emptyset$.

Therefore, in [8] we extend the theory of this cohomology to the case when $\partial M \neq \emptyset$. Through present variety decompositions of the space of $\Omega^k(M)$ in terms of the Morse-Novikov differential operator and its adjoint, we gain geometric and topological insights about Morse-Novikov cohomology groups. Furthermore, the deep relationship between the decompositions of $\Omega^k(M)$ and the topology and geometry of the refinement version of this cohomology group of manifold with boundary is also investigated in [9]. Here, we summarize the main results of [8] as follows:

Morse-Novikov cohomology with $\partial M \neq \emptyset$. Because of the Riemannian structure of M, on each $\alpha, \beta \in \Omega^k$, a natural inner product may be defined, and it is given by

$$\langle \alpha,\ \beta \rangle = \int_M \alpha \wedge (\star\beta), \tag{1.1}$$

where $\star : \Omega^k \to \Omega^{n-k}$ is the Hodge star operator [10]. Green's formula for d_θ and δ_θ is given by the following proposition:

Proposition 1.2 ([8]): We have

$$\langle \mathrm{d}_\theta \alpha, \beta \rangle = \langle \alpha, \delta_\theta \beta \rangle + \int_{\partial M} i^*(\alpha \wedge \star\beta), \tag{1.2}$$

where $\alpha \in \mathrm{H}^1\Omega^k$, $\beta \in \mathrm{H}^1\Omega^{k+1}$, $\delta_\theta = (-1)^{nk+1} \star \mathrm{d}_{-\theta} \star = \delta + (-1)^{nk} \star (\theta \wedge \star)$, and $i : \partial M \to M$ is the inclusion map.

The absolute and relative Morse-Novikov cohomology groups are defined by $H^k_\theta(M) = H^k(\Omega^*, \mathrm{d}_\theta)$ and $H^k_\theta(M,\ \partial M) = H^k(\Omega^*_D, \mathrm{d}_\theta)$, respectively. Where $\Omega^k_D = \{\xi \in \Omega^k(M) \mid i^*(\xi) = 0\}$ refers to the space of *Dirichlet* k-forms while $\Omega^k_N = \{\xi \in \Omega^k(M) \mid i^*(\star\xi) = 0\}$ refers to the space of *Neumann* k-forms. Moreover, the Morse-Novikov Laplacian operator is defined by $\Delta_\theta = \mathrm{d}_\theta\delta_\theta + \delta_\theta\mathrm{d}_\theta$, while the space of θ-harmonic fields $\mathcal{H}^k_\theta$ is $\mathcal{H}^k_\theta = \ker \delta_\theta \cap \ker \mathrm{d}_\theta$. $\mathcal{H}^k_\theta$ is infinite-dimensional. Consequently, it is far too vast to transmit information about cohomology. Therefore, we present the finite-dimensional subspaces of $\mathcal{H}^k_\theta$ which are

$$\mathcal{H}^k_{\theta,D} = \mathcal{H}^k_\theta \cap \Omega^k_D$$

and

$$\mathcal{H}^k_{\theta,N} = \mathcal{H}^k_\theta \cap \Omega^k_N$$

Our main results imply new decompositions to smooth forms Ω^k (or $L^2\Omega^k$):

Theorem 1.3 ([8]): *We have*

$$L^2\Omega^k = d_\theta\Omega^{k-1}_D \oplus \delta_\theta\Omega^{k+1}_N \oplus L^2\mathcal{H}^k_\theta$$

where $L^2\mathcal{H}^k_\theta(M) = \overline{\mathcal{H}^k_\theta}$.

Theorem 1.4 ([8]): *The space* $\mathcal{H}^k_\theta \subseteq \mathrm{H}^1\Omega^k(M)$ *can be written as:*

$$\mathcal{H}^k_\theta = \mathcal{H}^k_{\theta,\mathrm{co}} \oplus \mathcal{H}^k_{\theta,D},$$
$$\mathcal{H}^k_\theta = \mathcal{H}^k_{\theta,\mathrm{ex}} \oplus \mathcal{H}^k_{\theta,N},$$

such that $\mathcal{H}^k_{\theta,\mathrm{co}} = \{\rho \in \mathcal{H}^k_\theta \mid \rho = \delta_\theta\mu\}$ *and* $\mathcal{H}^k_{\theta,\mathrm{ex}} = \{\varsigma \in \mathcal{H}^k_\theta \mid \varsigma = d_\theta\chi\}$

Theorems 1.3 and 1.4 imply:

Corollary 1.5 ([8]): *We have*

$$L^2\Omega^k = d_\theta\Omega^{k-1}_D \oplus \delta_\theta\Omega^{k+1}_N \oplus \mathcal{H}^k_{\theta,D} \oplus L^2\mathcal{H}^k_{\theta,\mathrm{co}}, \tag{1.3}$$

$$L^2\Omega^k = d_\theta\Omega^{k-1}_D \oplus \delta_\theta\Omega^{k+1}_N \oplus \mathcal{H}^k_{\theta,N} \oplus L^2\mathcal{H}^k_{\theta,\mathrm{ex}}. \tag{1.4}$$

Consequently, we prove the following isomorphisms:

Theorem 1.6 ([8]): *For any closed one-form θ, we have*

1. $H^k_\theta(M, \partial M) \cong \mathcal{H}^k_{\theta,D}$.
2. $H^k_\theta(M) \cong \mathcal{H}^k_{\theta,N}$.
3. *(Poincaré-Lefschetz duality of Morse-Novikov): The Hodge $\star$ operator on Ω^k gives*

   $$H^k_\theta(M) \cong H^{n-k}_\theta(M, \partial M).$$

4. *(generalized Haller's isomorphism [11]):* $H^{n-k}_\theta(M, \partial M) \cong H^k_{-\theta}(M)$ *where* $H^k_{-\theta}(M) = H^k(\Omega^*, d_{-\theta})$.

On the other hand, the investigation of boundary value problems for differential forms on a compact Riemannian manifold with a boundary is not novel. Many researchers have studied boundary value problems of classical exterior derivative for differential forms in the literature, such as Duff and Spencer [12, 13] and [14]. In [15], Georgescu considers a variety of nonhomogeneous boundary value problems of exterior derivative for differential forms on a compact Riemannian manifold with boundary, and these problems considered are essentially the same as those considered by Duff and Spencer [13], where the methods he used were based on trace theorems and transposition. Bolik [16] gives solutions to boundary value problems for differential forms, generalizing Poisson's equation for functions. In [17, 18], Bolik provides solutions to second-order boundary value problems for differential forms by means of boundary integral methods, and further constraints are imposed by the boundary conditions and topological properties. Also, different kind of researches may be found in [19–22].

To the best of our knowledge, we have not encountered any work that discusses the relationship between Morse-Novikov cohomology groups and boundary value problems. As a result, this study attempts to move in this new direction.

First of all, we have noticed that under very restrictive conditions, Theorem 1.1 implicitly provides the sufficient condition of the existence solution $\omega \in \Omega^{k-1}(M)$ such that $\mathrm{d}_\theta \omega = \eta$ for any trivial Morse-Novicov cohomology class $[\eta] = [0]$. So, this motivates us to investigate the integrability requirements for such an equation without any restrictions on the manifold M with a nonempty boundary. To this end, we shall revisit the above decompositions of $\Omega^k(M)$, but in the context of solving differential equations on $\Omega^k(M)$ to answer the main question imposed in the abstract. Fortunately, the answer to this question is affirmative, and this led us to investigate the integrability conditions for a variety of perturbed boundary value problems of differential forms in terms of d_θ, δ_θ, and Δ_θ. We will do so by using the technique of the decompositions given above. Henceforth, M will be a compact, oriented, smooth Riemannian manifold with nonempty boundary.

2. The perturbed Dirichlet and Neumann boundary value problems

Theorem 2.1: *Suppose $\varsigma \in \Omega^{k+1}(M)$ and $\tau \in \Omega^k(\partial M)$, the perturbed Dirichlet problem*

$$\begin{cases} d_\theta \omega = \varsigma & \text{on } M, \\ i^*\omega = \tau & \text{on } \partial M \end{cases} \tag{2.1}$$

is solvable, iff ς and τ satisfy the integrability conditions

$$\langle \varsigma, \delta_\theta \beta \rangle = 0 \quad \forall \delta_\theta \beta \in \delta_\theta \Omega_N^{k+2} \tag{2.2}$$

$$\langle \varsigma, \kappa \rangle = \int_{\partial M} i^*(\omega \wedge \star\kappa)$$

$$= \int_{\partial M} \tau \wedge i^* \star \kappa \quad \forall \kappa \in \mathcal{H}_\theta^{k+1}(M). \tag{2.3}$$

A solution ω can be selected, such that $[\omega] \in H^k(\Omega^k(M), \delta_\theta)$.

Proof: Obviously, if Eq. (2.1) is given, then the integrability conditions (2.2) and (2.3) are satisfied. Now, to prove that these conditions are also sufficient, we apply Theorem 1.3 to ς, and obtain

$$\varsigma = \mathrm{d}_\theta \alpha_\varsigma + \delta_\theta \beta_\varsigma + \kappa_\varsigma.$$

The integrability condition (2.2) forces $\delta_\theta \beta_\varsigma = 0$. However, we can build an extension $\overline{\tau} \in \Omega^k(M)$ of τ where $i^*\overline{\tau} = \tau$ and $\overline{\tau}$ can be written as

$$\overline{\tau} = \delta_\theta \beta_{\overline{\tau}} + \kappa_{\overline{\tau}}. \tag{2.4}$$

We can neglect $\mathrm{d}_\theta \alpha_{\overline{\tau}}$ because $i^*\overline{\tau} = \tau$ and $i^*\mathrm{d}_\theta \alpha_{\overline{\tau}} = \mathrm{d}_{i^*\theta} i^* \alpha_{\overline{\tau}} = 0$. Now, using Theorem 1.3 again on $\eta = \mathrm{d}_\theta \overline{\tau}$ to get

$$\eta = \mathrm{d}_\theta \alpha_\eta + \kappa_\eta,$$

since η is d_θ-exact then $\delta_\theta \beta_\eta$ has to vanish. According to these decompositions, we set

$$\omega = \alpha_\varsigma + \overline{\tau} - \alpha_\eta.$$

Obviously, $\mathrm{d}_\theta \omega = \varsigma + \kappa_\eta - \kappa_\varsigma$ on M and $i^*\omega = \tau$ on ∂M. But Proposition 1.2 and integrability condition (2.3) assert that

$$\langle \kappa_\eta - \kappa_\varsigma, \lambda \rangle = \int_{\partial M} i^*(\omega \wedge \star\lambda) - \langle \varsigma, \lambda \rangle = 0$$
$$\forall \lambda \in \mathcal{H}_\theta^{k+1}(M).$$

Since, $\kappa_\eta - \kappa_\varsigma \in \mathcal{H}_\theta^{k+1}(M)$ then $\kappa_\eta - \kappa_\varsigma = 0$. Thus, there exists a solution $\omega \in \Omega^k(M)$ that satisfies Eq. (2.1). Furthermore,

$$\delta_\theta \omega = \delta_\theta \alpha_\varsigma + \delta_\theta \overline{\tau} - \delta_\theta \alpha_\eta = 0$$

because the components α_ς and α_η can be chosen such that $\delta_\theta \alpha_\varsigma = \delta_\theta \alpha_\eta = 0$ (using the principle of gauge freedom in our decompositions), and since $\delta_\theta \overline{\tau} = 0$, by Eq. (2.4), we obtain $[\omega] \in H^k(\Omega^k(M), \delta_\theta)$. □

Corollary 2.2: *The integrability projection conditions (2.2) and (2.3) for the perturbed Dirichlet boundary* *value problem (2.1) are equivalent to the following differential conditions:*

$$d_\theta \varsigma = 0, \quad i^*\varsigma = d_{i^*\theta}\tau \tag{2.5}$$

$$\langle \varsigma, \lambda\rangle = \int_{\partial M} \tau \wedge i^* \star \lambda,$$
$$\forall \lambda \in \mathcal{H}^{k+1}_{\theta,D}(M) \cong H^{k+1}_\theta(M, \partial M). \tag{2.6}$$

Proof: We notice that Proposition 1.2 implies the equivalence between condition (2.2) and

$$\langle \mathrm{d}_\theta \varsigma, \beta\rangle = 0 \quad \forall \beta \in \Omega^{k+2}_N(M),$$

since, $\Omega^{k+2}_N(M) \subseteq L^2\Omega^{k+2}(M)$ is dense, condition (2.2) is equivalence to $\mathrm{d}_\theta\varsigma = 0$. The construction of the solution ω in the proof of Theorem 2.1 and condition (2.3) induce that

$$\langle \varsigma, \kappa\rangle = \int_{\partial M} \tau \wedge i^* \star \kappa = \langle \mathrm{d}_\theta \overline{\tau}, \kappa\rangle \quad \forall \kappa \in \mathcal{H}^{k+1}_\theta(M),$$

where $i^*\varsigma = \mathrm{d}_{i^*\theta}\tau$, and $i^*\overline{\tau} = \tau$. It means that condition (2.6) is a subcondition of (2.3). Thus, this proves conditions (2.2) and (2.3) imply conditions (2.5) and (2.6).

In turn, Theorem 1.4 asserts that $\kappa \in \mathcal{H}^{k+1}_\theta(M)$ splits orthogonally into $\kappa = \lambda_\kappa + \delta_\theta\gamma_\kappa$ where $\lambda_\kappa \in \mathcal{H}^{k+1}_{\theta,D}(M)$ and $\delta_\theta\gamma_\kappa \in \mathcal{H}^{k+1}_{\theta,co}(M)$. Hence, condition (2.3) is equivalent to the following two independent conditions: the first one is in fact the integrability condition (2.6), while the other one is:

$$\langle \varsigma, \delta_\theta\gamma_\kappa\rangle = \int_{\partial M} \tau \wedge i^* \star \delta_\theta\gamma_\kappa \quad \forall \delta_\theta\gamma_\kappa \in \mathcal{H}^{k+1}_{\theta,co}(M). \tag{2.7}$$

We must now demonstrate that Eq. (2.7) follows from (2.5). But, Proposition 1.2 and (2.5) imply

$$\langle \varsigma, \delta_\theta\gamma_\kappa\rangle = -\int_{\partial M} i^*(\varsigma \wedge \star\gamma_\kappa) = -\int_{\partial M} \mathrm{d}_{i^*\theta}\tau \wedge i^* \star \gamma_\kappa$$
$$= \int_{\partial M} \tau \wedge i^* \star \delta_\theta\gamma_\kappa.$$

This identity shows that condition (2.7) follows from (2.5). Hence, this proves the converse. □

Theorem 2.3: *Suppose $\sigma \in \Omega^{k-1}(M)$ and $\varphi \in \Omega^{n-k}(\partial M)$, the perturbed Neumann problem*

$$\begin{cases} \delta_\theta\omega &= \sigma \quad \text{on} \quad M, \\ i^*\star\omega &= \varphi \quad \text{on} \quad \partial M, \end{cases} \tag{2.8}$$

is solvable iff σ and φ satisfy the integrability conditions

$$\langle \sigma, d_\theta\alpha\rangle = 0 \quad \forall d_\theta\alpha \in d_\theta\Omega^{k-2}_D \tag{2.9}$$

$$\langle \sigma, \lambda\rangle = -\int_{\partial M} i^*(\lambda \wedge \star\omega)$$
$$= -\int_{\partial M} i^*\lambda \wedge \varphi, \quad \forall \lambda \in \mathcal{H}^{k-1}_\theta(M). \tag{2.10}$$

A solution ω can be selected, such that $[\omega] \in H^k_\theta(M)$.

Proof: We set $\mu = \star\omega \in \Omega^{n-k}(M)$, problem (2.8) is then equivalent to the following Dirichlet problem:

$$\mathrm{d}_\theta\mu = (-1)^k \star \sigma \quad i^*\mu = \varphi,$$

which by Theorem 2.1 is solvable iff $\star\sigma$ and φ satisfy the integrality conditions

$$\langle \star\sigma, \delta_\theta\beta\rangle = 0 \quad \forall \delta_\theta\beta \in \delta_\theta\Omega^{n-k+2}_N(M)$$
$$\langle \star\sigma, \kappa\rangle = (-1)^k \int_{\partial M} \varphi \wedge i^* \star \kappa \quad \forall \kappa \in \mathcal{H}^{n-k+1}_\theta(M).$$

Now, if we set $\lambda = \star\kappa$ and $\alpha = \star\beta$, then these conditions are equivalent respectively to (2.9) and (2.10). Since, μ can be chosen, such that $\delta_\theta\mu = 0$, then this implies ω can be chosen, such that $[\omega] \in H^k_\theta(M)$ as required. □

Corollary 2.4: *The integrability projection conditions (2.9) and (2.10) for the perturbed Neumann boundary value problem (2.8) are equivalent to the following differential conditions*

$$\delta_\theta\sigma = 0, \quad i^* \star \sigma = (-1)^k d_{i^*\theta}\varphi \tag{2.11}$$

$$\langle \sigma, \lambda\rangle = -\int_{\partial M} i^*\lambda \wedge \varphi, \quad \forall \lambda \in \mathcal{H}^{k-1}_{\theta,N}(M) \cong H^{k-1}_\theta(M). \tag{2.12}$$

Proof: Similarly, using the same setting as above, the proof will follow from Corollary 2.2. □

Remark 2.5: In applications, the differential condition (2.6) has the important feature of allowing one to assess the integral $\langle \varsigma, \lambda\rangle = \int_{\partial M} \tau \wedge i^* \star \lambda$ only for a finite number of $\mathcal{H}^{k+1}_{\theta,D}(M)$, whereas the projection condition (2.3) would demand a comparison of these integrals for every infinite dimensional $\mathcal{H}^{k+1}_\theta(M)$. Similarly, for the perturbed Neumaan problem (2.8).

3. Nonvanishing Morse-Novikov cohomology group

Firstly, we have to exclude another constraint on the manifold M with boundary rather than to be an almost nilpotent closed manifold (see, Theorem 1.1), which make its Morse-Novikov cohomology group vanish.

Lemma 3.1 (The Poincaré lemma for Morse-Novikov cohomology): *Let M be a contractible manifold, then Morse-Novikov cohomology groups $H^k_\theta(M)$ and $H^k_\theta(M, \partial M)$ vanish for all k, where $0 \neq [\theta] \in H^1_{dR}(M)$.*

Proof: In [8], Theorem 26 asserts that "If $[\theta_1], [\theta_2] \in H^1_{dR}(M)$ such that $[\theta_1] = [\theta_2]$ then $H^k_{\theta_1}(M, \partial M) \cong H^k_{\theta_2}(M, \partial M)$, and $H^k_{\theta_1}(M) \cong H^k_{\theta_2}(M)$, $\forall k \geq 0$". Since, M is contractible then $H^k_{dR}(M) = 0$ and thus $[\theta] = [0]$ (by Poincaré lemma of dr Rham cohomology). Therefore, Theorem 26 implies that $H^k_\theta(M) \cong H^k_0(M) = H^k_{dR}(M) = 0$ as required. Moreover, Poincaré-Lefschetz duality of Morse-Novikov (Theorem 1.6) implies $H^k_\theta(M, \partial M)$ vanishes as well.□

Henceforth, M will be a noncontractible compact, oriented smooth Riemannian manifold with boundary and a nonvanishing Morse-Novikov cohomology group. The following theorems provide the necessarily and sufficient conditions of the existence solution of $\mathrm{d}_\theta \omega = \eta$ for $[\eta] \in H^k_\theta(M)$ and for $[\eta] \in H^k_\theta(M, \partial M)$, respectively.

Theorem 3.2: *Let $\xi \in \Omega^k(M)$. Then $\xi \in d_\theta \Omega^{k-1}(M)$ (i.e. ξ is $d_\theta-$exact) if and only if ξ obeys the integrability conditions*

$$[\xi] \in H^k_\theta(M) \quad \text{and} \quad \langle \xi, \lambda \rangle = 0 \quad \forall \lambda \in \mathcal{H}^k_{\theta,N}. \tag{3.1}$$

Proof: Let $\xi \in \mathrm{d}_\theta \Omega^{k-1}(M)$ then there exists a solution $\omega \in \Omega^{k-1}(M)$ such that $\xi = \mathrm{d}_\theta \omega$. Using Eq. (1.2), one can easily verify that ξ satisfies conditions (3.1).

Now, to prove the converse: Eq. (1.4) in Corollary 1.5 implies

$$\xi = \mathrm{d}_\theta \alpha_\xi + \delta_\theta \beta_\xi + \lambda_\xi + \mathrm{d}_\theta \varepsilon_\xi \in \mathrm{d}_\theta \Omega^{k-1}_D \oplus \delta_\theta \Omega^{k+1}_N \oplus \mathcal{H}^k_{\theta,N}(M) \oplus L^2\mathcal{H}^k_{\theta,\mathrm{ex}}(M).$$

We infer $\delta_\theta \beta_\xi = 0$ and $\lambda_\xi = 0$ because conditions (3.1) and Eq. (1.2) imply that $\langle \mathrm{d}_\theta \xi, \beta_\xi \rangle = \langle \xi, \delta_\theta \beta_\xi \rangle = \|\delta_\theta \beta_\xi\|^2 = 0$ and $\langle \xi, \lambda_\xi \rangle = \|\lambda_\xi\|^2 = 0$, respectively. Thus, there exists $\omega = \alpha_\xi + \varepsilon_\xi \in \Omega^{k-1}(M)$ such that $\xi = \mathrm{d}_\theta \omega$ which demonstrates that conditions (3.1) are equivalent to $\xi \in \mathrm{d}_\theta \Omega^{k-1}(M)$. □

Now, we have the following theorem for the necessary and sufficient conditions of the existence solution for the relative case:

Theorem 3.3: *Let $\xi \in \Omega^k_D(M)$. Then $\xi \in d_\theta \Omega^{k-1}_D(M)$ iff ξ satisfies the integrability conditions*

$$[\xi] \in H^k_\theta(M, \partial M) \quad \text{and} \quad \langle \xi, \lambda \rangle = 0 \quad \forall \lambda \in \mathcal{H}^k_{\theta,D}. \tag{3.2}$$

Proof: As above, it is easy to verify condition (3.2) when $\xi \in \mathrm{d}_\theta \Omega^{k-1}_D(M)$ such that $i^*\xi = 0$. Conversely, Corollary 1.5 implies

$$\xi = \mathrm{d}_\theta \zeta_\xi + \delta_\theta \gamma_\xi + \kappa_\xi + \delta_\theta \varepsilon_\xi \in \mathrm{d}_\theta \Omega^{k-1}_D \oplus \delta_\theta \Omega^{k+1}_N \oplus \mathcal{H}^k_{\theta,D}(M) \oplus L^2\mathcal{H}^k_{\theta,\mathrm{co}}(M).$$

In this case, we must have $\delta_\theta \gamma_\xi = 0$, $\kappa_\xi = 0$ and $\delta_\theta \varepsilon_\xi = 0$ because conditions (3.2) and Eq. (1.2) imply that $\langle \mathrm{d}_\theta \xi, \gamma_\xi \rangle = \langle \xi, \delta_\theta \gamma_\xi \rangle = \|\delta_\theta \gamma_\xi\|^2 = 0$, $\langle \xi, \kappa_\xi \rangle = \|\kappa_\xi\|^2 = 0$ and $\langle \mathrm{d}_\theta \xi, \varepsilon_\xi \rangle = \langle \xi, \delta_\theta \varepsilon_\xi \rangle = \|\delta_\theta \varepsilon_\xi\|^2 = 0$, respectively. Thus, there exists $\zeta_\xi \in \Omega^{k-1}_D(M)$ such that $\xi = \mathrm{d}_\theta \zeta_\xi$ which proves that conditions (3.2) are equivalent to $\xi \in \mathrm{d}_\theta \Omega^{k-1}_D(M)$. □

Corollary 3.4: *Let $\zeta \in \Omega^k(M)$. Then $\zeta \in \delta_\theta \Omega^{k+1}(M)$ (i.e. ζ is $\delta_\theta-$coexact) iff ζ satsfies the integrability conditions*

$$[\zeta] \in H^k(\Omega^k(M), \delta_\theta) \quad \text{and} \quad \langle \zeta, \kappa \rangle = 0 \quad \forall \kappa \in \mathcal{H}^k_{\theta,D}. \tag{3.3}$$

Proof: Similarly, it is easy to verify condition (3.3) when $\zeta \in \delta_\theta \Omega^{k+1}(M)$. Now to prove the converse, Corollary 1.5 implies

$$\zeta = \mathrm{d}_\theta \xi_\zeta + \delta_\theta \gamma_\zeta + \kappa_\zeta + \delta_\theta \varepsilon_\zeta \in \mathrm{d}_\theta \Omega^{k-1}_D \oplus \delta_\theta \Omega^{k+1}_N \oplus \mathcal{H}^k_{\theta,D}(M) \oplus L^2\mathcal{H}^k_{\theta,\mathrm{co}}(M).$$

In this case, we must have $\mathrm{d}_\theta \xi_\zeta = 0$, and $\kappa_\zeta = 0$ because condition (3.3) and Eq. (1.2) imply that $\langle \xi_\zeta, \delta_\theta \zeta \rangle = \langle \mathrm{d}_\theta \xi_\zeta, \zeta \rangle = \|\mathrm{d}_\theta \xi_\zeta\|^2 = 0$, and $\langle \zeta, \kappa_\zeta \rangle = \|\kappa_\zeta\|^2 = 0$, respectively. Hence, there exists $\eta = \gamma_\zeta + \varepsilon_\zeta$, such that $\zeta = \delta_\theta \eta$ which illustrate that conditions (3.3) are equivalent to $\zeta \in \delta_\theta \Omega^{k+1}(M)$. □

Theorem 3.5: *Let $\xi \in \Omega^{k+1}(M)$, $\rho \in \Omega^{k-1}(M)$ and $\vartheta \in \Omega^k(\partial M)$, the boundary value problem*

$$\begin{cases} d_\theta \mu &= \xi \quad \text{and} \quad \delta_\theta \mu = \rho \quad \text{on} \quad M, \\ i^*\mu &= \vartheta \quad \text{on} \quad \partial M, \end{cases} \tag{3.4}$$

is solvable, if and only if

$$[\rho] \in H^{k-1}(\Omega^k(M), \delta_\theta), \quad \langle \rho, \kappa \rangle = 0 \quad \forall \kappa \in \mathcal{H}^{k-1}_{\theta,D}$$

and

$$[\xi] \in H^{k+1}_\theta(M), \quad i^*\xi = d_{i^*\theta}\vartheta,$$

$$\langle \xi, \lambda \rangle = \int_{\partial M} i^*\mu \wedge i^* \star \lambda, \quad \forall \lambda \in \mathcal{H}^{k+1}_{\theta,D}$$

Uniqueness of the solution of (3.4) is determined by arbitrary Dirichlet θ-harmonic fields $\mathcal{H}^k_{\theta,D}$.

Proof: If Eq. (3.4) is given, then it satisfies directly these conditions. Now, the supposed integrability conditions on ρ together with Corollary 3.4 imply the existence of μ_ρ, such that $\rho = \delta_\theta \mu_\rho$. Hence, we set $\mu = \mu_\xi + \mu_\rho$, so Eq. (3.4) turns into the following Dirichlet problem of d_θ,

$$\mathsf{d}_\theta \mu_\xi = \xi - \mathsf{d}_\theta \mu_\rho \quad \textit{and} \quad \delta_\theta \mu_\xi = 0 \quad \text{on} \quad M$$

$$i^* \mu_\xi = \vartheta - i^* \mu_\rho \quad \text{on} \quad \partial M$$

which is solvable by Theorem 2.1 together with Corollary 2.2. More precisely, conditions (2.5) and (2.6) for this problem are equivalent to

$$\mathsf{d}_\theta(\mathsf{d}_\theta \mu_\rho) = 0, \quad i^*(\mathsf{d}_\theta \mu_\rho) = \mathsf{d}_{i^*\theta}(i^* \mu_\rho) \quad \textit{and}$$

$$\langle \mathsf{d}_\theta \mu_\rho, \lambda \rangle = \int_{\partial M} i^* \mu_\rho \wedge i^* \star \lambda, \quad \forall \lambda \in \mathcal{H}^{k+1}_{\theta,D}.$$

Thus, there exists a solution $\mu = \mu_\xi + \mu_\rho$ to Eq. (3.4). □

Furthermore, using the duality given in Section 2 between d_θ and δ_θ, we infer:

Corollary 3.6: *Let $\vartheta \in \Omega^k(\partial M)$. The boundary value problem*

$$\begin{cases} \mathsf{d}_\theta \mu &= \xi \quad \textit{and} \quad \delta_\theta \mu = \rho \quad \textit{on} \quad M, \\ i^* \star \mu &= \star \vartheta \quad \textit{on} \quad \partial M, \end{cases} \tag{3.5}$$

is solvable, if and only if

$$[\xi] \in H^{k+1}_\theta(M), \quad \langle \xi, \kappa \rangle = 0 \quad \forall \kappa \in \mathcal{H}^{k+1}_{\theta,N}$$

and

$$[\rho] \in H^{k-1}(\Omega^k(M), \delta_\theta), \quad i^*(\star \rho) = (\pm) \star \delta_{i^*\theta} \vartheta,$$

$$\langle \rho, \lambda \rangle = -\int_{\partial M} i^* \lambda \wedge i^* \star \mu, \quad \forall \lambda \in \mathcal{H}^{k-1}_{\theta,N}.$$

The uniqueness solution of (3.5) is determined by arbitrary Neumann θ-harmonic fields $\mathcal{H}^k_{\theta,N}$.

Proposition 3.7: Let $\vartheta \in \Omega^k(\partial M)$.

1. There exists a $\theta-$harmonic field $\kappa_\theta \in \mathcal{H}^k_\theta(M)$ satisfying $i^* \kappa_\theta = \vartheta$ if and only if

$$\mathsf{d}_{i^*\theta} \vartheta = 0, \quad \int_{\partial M} \vartheta \wedge i^* \star \lambda = 0, \quad \forall \lambda \in \mathcal{H}^{k+1}_{\theta,D}. \tag{3.6}$$

2. There exists a $\theta-$harmonic field $\gamma_\theta \in \mathcal{H}^k_\theta(M)$ satisfying $i^* \star \gamma_\theta = \star \vartheta$ if and only if

$$\delta_{i^*\theta} \vartheta = 0, \quad \int_{\partial M} i^* \lambda \wedge \star \vartheta = 0, \quad \forall \lambda \in \mathcal{H}^{k-1}_{\theta,N}. \tag{3.7}$$

Proof: Branch (1) is a direct consequence of Theorem 3.5, since $\kappa_\theta \in \mathcal{H}^k_\theta(M)$ then the boundary value problem reduce to

$$\mathsf{d}_\theta \kappa_\theta = 0, \quad \delta_\theta \kappa_\theta = 0 \quad \textit{and} \quad i^* \kappa_\theta = \vartheta$$

which is solvable iff conditions (3.6) holds. Similarly, the dual results of branch (2) follow from branch (1) together with the fact that $\kappa_\theta \in \mathcal{H}^k_\theta(M)$ iff $\star \kappa_\theta \in \mathcal{H}^{n-k}_\theta(M)$. □

Proposition 3.7 gives us the following interesting decomposition.

Proposition 3.8: The pullback of the space of $\theta-$harmonic filed $i^* \mathcal{H}^k_\theta(M)$ can be decomposed into:

$$i^* \mathcal{H}^k_\theta(M) = \mathcal{E}^k_{i^*\theta}(\partial M) + i^* \mathcal{H}^k_{\theta,N}(M).$$

Where $i^* \mathcal{H}^k_{\theta,N}(M) = \{i^* \kappa \mid \kappa \in \mathcal{H}^k_{\theta,N}(M)\}$ and $\mathcal{E}^k_{i^*\theta}(\partial M) = \{\mathsf{d}_{i^*\theta} \gamma \mid \gamma \in \Omega^{k-1}(\partial M)\}$.

Proof: Let $\kappa \in \mathcal{H}^k_\theta(M)$ then it can be written in the form $\kappa = \mathsf{d}_\theta \gamma + \nu \in \mathcal{H}^k_{\theta,\text{ex}}(M) \oplus \mathcal{H}^k_{\theta,N}(M)$ by Theorem 1.4. Clearly, this gives

$$i^* \kappa = \mathsf{d}_{i^*\theta} i^* \gamma + i^* \nu \in \mathcal{E}^k_{i^*\theta}(\partial M) + i^* \mathcal{H}^k_{\theta,N}(M).$$

So, this proves that $i^* \mathcal{H}^k_\theta(M) \subseteq \mathcal{E}^k_{i^*\theta}(\partial M) + i^* \mathcal{H}^k_{\theta,N}(M)$. Conversely, for $\mathsf{d}_{i^*\theta} \vartheta \in \Omega^k(\partial M)$ and $\lambda \in \mathcal{H}^{k+1}_{\theta,D}(M)$, we get

$$\int_{\partial M} \mathsf{d}_{i^*\theta} \vartheta \wedge i^* \star \lambda = \int_{\partial M} \mathsf{d}_{i^*\theta}(\vartheta \wedge i^* \star \lambda) = 0.$$

Hence, $\mathsf{d}_{i^*\theta} \vartheta$ satisfies

$$\mathsf{d}_{i^*\theta}(\mathsf{d}_{i^*\theta} \vartheta) = 0, \quad \int_{\partial M} \mathsf{d}_{i^*\theta} \vartheta \wedge i^* \star \lambda = 0$$
$$\forall \lambda \in \mathcal{H}^{k+1}_{\theta,D}(M)$$

which is in fact condition (3.6). So, Proposition 3.7 implies that there exists $\kappa \in \mathcal{H}^k_\theta(M)$ such that $i^* \kappa = \mathsf{d}_{i^*\theta} \vartheta$. This proves the converse is true as well. □

Proposition 3.9: The space $\mathcal{H}^k_{\theta,N}$ (or $\mathcal{H}^k_{\theta,D}$) can be determined uniquely by the space $i^* \mathcal{H}^k_{\theta,N}$ (or

$i^*\mathcal{H}^k_{\theta,D}$) respectively. Consequently, $i^*\mathcal{H}^k_{\theta,N} \cong H^k_\theta(M)$ and $i^*\mathcal{H}^k_{\theta,D} \cong H^k_\theta(M, \partial M)$.

Proof: We merely need to demonstrate this

$$\mathcal{H}^k_{\theta,N} \cong i^*\mathcal{H}^k_{\theta,N}.$$

Clearly, $i^* : \mathcal{H}^k_{\theta,N} \to i^*\mathcal{H}^k_{\theta,N}$ is surjective and also it is injective because $\ker i^* = \{0\}$, so it is a bijection map. Hence, $\mathcal{H}^k_{\theta,N} \cong i^*\mathcal{H}^k_{\theta,N} \cong H^k_\theta(M)$. Consequently, combining this with Theorem 1.6 (Morse-Novikov-Poincaré-Lefschetz duality), we infer that $i^*\mathcal{H}^{n-k}_{\theta,N} \cong \mathcal{H}^k_{\theta,D} \cong i^*\mathcal{H}^k_{\theta,D}$ and hence $i^*\mathcal{H}^k_{\theta,D} \cong H^k_\theta(M, \partial M)$. □

Theorem 3.10: *Given $\chi \in \Omega^k(M)$ and $\vartheta \in \Omega^k(\partial M)$. Then the mixed boundary value problem of perturbed Poisson equation*

$$\begin{cases} \Delta_\theta \mu & = \quad \chi \quad \text{on} \quad M, \\ i^*\mu & = \quad \vartheta \quad \text{and} \quad i^*\delta_\theta \mu = 0 \quad \text{on} \quad \partial M, \end{cases} \tag{3.8}$$

is solvable, if and only if

$$\langle \chi, \lambda \rangle = 0 \quad \forall \lambda \in \mathcal{H}^k_{\theta,D}. \tag{3.9}$$

The uniqueness solution of (3.8) is determined by an arbitrary Dirichlet θ-harmonic field.

Proof: Clearly, Green's formula for d_θ and δ_θ Eq. (1.2) implies that Eq. (3.8) can satisfy condition (3.9).

Now assume $\chi \in \Omega^k(M)$ satisfies $\langle \chi, \lambda \rangle = 0$, $\forall \lambda \in \mathcal{H}^k_{\theta,D}$ (i.e. $\chi \in (\mathcal{H}^k_{\theta,D})^\perp$). We can, however, build an extension $\mu_1 \in \Omega^k(M)$ to $\vartheta \in \Omega^k(\partial M)$ such that

$$i^*\mu_1 = \vartheta, \quad \mu_1 = \delta_\theta \beta_{\mu_1} + \kappa_{\mu_1} \in \delta_\theta \Omega^{k+1}_N(M) \oplus \mathcal{H}^k_\theta(M).$$

We are able to do so, the component $\mathrm{d}_\theta \alpha_{\mu_1} \in \mathrm{d}_\theta \Omega^{k-1}_D(M)$ of an arbitrary extension μ_1 makes no contribution to the portion $i^*\mu_1$ because of $i^*\alpha_{\mu_1} = 0$. Now, Eq. (1.2) implies that $\langle \Delta_\theta \mu_1, \lambda \rangle = 0$, $\forall \lambda \in \mathcal{H}^k_{\theta,D}$ which means $\Delta_\theta \mu_1 \in (\mathcal{H}^k_{\theta,D})^\perp$ as well. Hence, $\chi - \Delta_\theta \mu_1 \in (\mathcal{H}^k_{\theta,D})^\perp$. We are now implementing Proposition 13 in [8], since $\chi - \Delta_\theta \mu_1 \in (\mathcal{H}^k_{\theta,D})^\perp$ is smooth, it follows that there is a unique smooth differential form $\mu_2 \in \Omega^k_D \cap (\mathcal{H}^k_{\theta,D})^\perp$ which satisfies the equation

$$\begin{cases} \Delta_\theta \mu_2 & = \quad \chi - \Delta_\theta \mu_1 \quad \text{on} \quad M, \\ i^*\mu_2 & = \quad 0 \quad \text{on} \quad \partial M, \\ i^*(\delta_\theta \mu_2) & = \quad 0 \quad \text{on} \quad \partial M. \end{cases} \tag{3.10}$$

Now, let $\mu_2 = \mu - \mu_1$, then Eq. (3.10) becomes an Eq. (3.8). Hence, there is a solution to the perturbed Poisson equation which is $\mu = \mu_1 + \mu_2$, where the uniqueness of μ is determined by an arbitrary Dirichlet θ-harmonic field. □

Now, we are going to present fruitful analysis for a special kind of perturbed Poisson Eq. (3.8), but with $\vartheta = 0$. So, we have the following results.

Proposition 3.11: The eigenvalues of the restricted Morse-Novikov Laplacian operator

$$\widetilde{\Delta}_\theta : \mathrm{H}^2\widetilde{\Omega}^k \longrightarrow L^2\Omega^k$$

are positive, where $\mathrm{H}^2\widetilde{\Omega}^k(M) = \{\nu \in \mathrm{H}^2\Omega^k \mid i^*\nu = 0,\ i^*(\delta_\theta \nu) = 0\}$.

Proof: Let $\eta \in \mathrm{H}^2\widetilde{\Omega}^k(M)$ and $\lambda \in \mathbb{R}$ such that $\widetilde{\Delta}_\theta \eta = \lambda \eta$. So, it follows that $\langle \widetilde{\Delta}_\theta \eta, \eta \rangle = \lambda \langle \eta, \eta \rangle$. But, Eq. (1.2) asserts that the left hand side can be written by the form $\|\mathrm{d}_\theta \eta\|^2 + \|\delta_\theta \eta\|^2 = \lambda \|\eta\|^2$. Therefore, λ must be positive. □

Proposition 3.12: Let $\lambda_1 \neq \lambda_2$ be two eigenvalues of $\widetilde{\Delta}_\theta$, then the eigenfunctions η and γ corresponding to λ_1 and λ_2 respectively, are L^2-orthogonal.

Proof: Since, $\eta, \gamma \in \mathrm{H}^2\widetilde{\Omega}^k(M)$ be eigenfunctions then we have $\widetilde{\Delta}_\theta \eta = \lambda_1 \eta$ and $\widetilde{\Delta}_\theta \gamma = \lambda_2 \gamma$. Clearly, $\widetilde{\Delta}_\theta$ is a self adjoint operator on $\mathrm{H}^2\widetilde{\Omega}^k(M)$, so it follows first that $\langle \widetilde{\Delta}_\theta \eta, \gamma \rangle = \langle \eta, \widetilde{\Delta}_\theta \gamma \rangle = \lambda_1 \langle \eta, \gamma \rangle$, but $\langle \eta, \widetilde{\Delta}_\theta \gamma \rangle = \lambda_2 \langle \eta, \gamma \rangle$. Combining all these together, we get $\lambda_1 \langle \eta, \gamma \rangle = \lambda_2 \langle \eta, \gamma \rangle$ which must imply that $\langle \eta, \gamma \rangle = 0$ as required because $\lambda_1 \neq \lambda_2$. □

Remark 3.13: Proposition 3.12 shows that if $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues of $\widetilde{\Delta}_\theta$ then the set of corresponding eigenfunctions $\{\eta_1, \eta_2, \ldots, \eta_m\}$ spans a subspace of the orthogonal complement of $\ker \widetilde{\Delta}_\theta$.

4. Future work

In the context of the Atiyah-Singer index theorem [23], the existence and uniqueness of solutions given in Theorem 3.5 could be understood as an existence result for the Dirichlet boundary value problem for a perturbed nonhomogenous Dirac operator, which we can define as

$$\mathrm{d}_\theta + \delta_\theta : \Omega^* \to \Omega^{even} \oplus \Omega^{odd}$$

where $\Omega^{even}(M)$ and $\Omega^{odd}(M)$ refer to differential forms of even and odd degree, respectively. So that $\Omega^* = \Omega^{even} \oplus \Omega^{odd}$. Furthermore, the Atiyah-Singer index theorem, asserts that the topological (analytical) index of every elliptic complex of M is independent. In

[8], we obtain that the Euler characteristics $\chi(M,\theta)$ of Morse-Novikov complex is given by

$$\begin{aligned}\chi(M,\theta) &= \chi(\partial M,\theta) + \chi(M,\partial M,\theta)\\ &= \chi(\partial M) + \chi(M,\partial M) = \chi(M).\end{aligned}$$

Hence, these results may correspond to a preliminary proposal to establish the perturbed generalized Atiyah-Singer index theorem on manifolds with boundary.

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Conflicts of interest

No conflicts exist.

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